## INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

**1.** Make use of the standard integrals above to find each indefinite integral.

(a) $\int \frac{1}{\sqrt{4-x^2}}dx$ (b) $\int \frac{1}{\sqrt{16-x^2}}dx$ (c) $\int \frac{1}{\sqrt{100-x^2}}dx$

(d) $\int \frac{1}{9+x^2}dx$ (e) $\int \frac{1}{25+x^2}dx$ (f) $\int \frac{1}{4+x^2}dx$

(g) $\int \frac{1}{\sqrt{25-x^2}}dx$ (h) $\int \frac{2}{\sqrt{1-x^2}}dx$ (i) $\int \frac{1}{36+x^2}dx$

(j) $\int \frac{2}{16+x^2}dx$ (k) $\int \frac{1}{\sqrt{49-x^2}}dx$ (l) $\int \frac{4}{100+x^2}dx$

(m) $\int \frac{1}{64+x^2}dx$ (n) $\int \frac{6}{x^2+9}dx$ (o) $\int \frac{1}{\sqrt{2-x^2}}dx$

(p) $\int \frac{1}{3+x^2}dx$ (q) $\int \frac{2}{2+x^2}dx$ (r) $\int \frac{1}{\sqrt{1-4x^2}}dx$

(s) $\int \frac{1}{1+9x^2}dx$ (t) $\int \frac{1}{\sqrt{9-4x^2}}dx$ (u) $\int \frac{1}{9+4x^2}dx$

(v) $\int \frac{1}{\sqrt{16-25x^2}}dx$ (w) $\int \frac{1}{16+9x^2}dx$ (x) $\int \frac{1}{\sqrt{100-9x^2}}dx$

**2.** Evaluate each definite integral in terms of $\pi$ :

(a) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx$ (b) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}dx$ (c) $\int_0^{3} \frac{1}{\sqrt{36-x^2}}dx$

(d) $\int_0^{1} \frac{1}{1+x^2}dx$ (e) $\int_0^{4} \frac{1}{16+x^2}dx$ (f) $\int_{\sqrt{3}}^{3} \frac{1}{9+x^2}dx$

(g) $\int_0^{4} \frac{1}{\sqrt{64-x^2}}dx$ (h) $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}dx$ (i) $\int_{-2}^{2} \frac{1}{4+x^2}dx$

(j) $\int_1^{\sqrt{3}} \frac{2}{1+x^2}dx$ (k) $\int_{-1}^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}dx$ (l) $\int_3^{6} \frac{4}{\sqrt{36-x^2}}dx$

(m) $\int_0^{1} \frac{1}{\sqrt{2-x^2}}dx$ (n) $\int_0^{\frac{3}{4}} \frac{1}{\sqrt{9-4x^2}}dx$

**3.** Show that $\int_1^{3} \frac{1}{3+x^2}dx = \frac{\pi}{6\sqrt{3}}$.

**4.** Use the substitution given in brackets to find each indefinite integral.

(a) $\int \frac{1}{x^2+2x+5}dx$ $(u = x+1)$ (b) $\int \frac{1}{x^2+4x+13}dx$ $(u = x+2)$.

(c) $\int \frac{1}{x^2-6x+10}dx$ $(u = x-3)$ (d) $\int \frac{1}{x^2-4x+5}dx$ $(u = x-2)$

(e) $\int \frac{1}{x^2+8x+20}dx$ $(u = x+4)$ (f) $\int \frac{1}{x^2-2x+3}dx$ $(u = x-1)$

**5.** Make use of the substitution $u = x-1$ to evaluate the definite integral $\int_1^{4} \frac{1}{x^2-2x+10}dx$, expressing your answer in terms of $\pi$.

## ANSWERS

1. (a) $\sin^{-1}\left(\frac{x}{2}\right)+C$ (b) $\sin^{-1}\left(\frac{x}{4}\right)+C$ (c) $\sin^{-1}\left(\frac{x}{10}\right)+C$

(d) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$ (e) $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$ (f) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

(g) $\sin^{-1}\left(\frac{x}{5}\right)+C$ (h) $2\sin^{-1}x+C$ (i) $\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+C$

(j) $\frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right)+C$ (k) $\sin^{-1}\left(\frac{x}{7}\right)+C$ (l) $\frac{2}{5}\tan^{-1}\left(\frac{x}{10}\right)+C$

(m) $\frac{1}{8}\tan^{-1}\left(\frac{x}{8}\right)+C$ (n) $2\tan^{-1}\left(\frac{x}{3}\right)+C$ (o) $\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

(p) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$ (q) $\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$ (r) $\frac{1}{2}\sin^{-1}(2x)+C$

(s) $\frac{1}{3}\tan^{-1}(3x)+C$ (t) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$ (u) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$

(v) $\frac{1}{5}\sin^{-1}\left(\frac{5x}{4}\right)+C$ (w) $\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)+C$ (x) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{10}\right)+C$

2. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{16}$ (f) $\frac{\pi}{36}$

(g) $\frac{\pi}{6}$ (h) $\frac{\pi}{6}$ (i) $\frac{\pi}{4}$ (j) $\frac{\pi}{6}$ (k) $\frac{5\pi}{12}$ (l) $\frac{4\pi}{3}$

(m) $\frac{\pi}{4}$ (n) $\frac{\pi}{12}$

4. (a) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$ (c) $\tan^{-1}(x-3)+C$

(d) $\tan^{-1}(x-2)+C$ (e) $\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+C$ (f) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

5. $\frac{\pi}{12}$

**INTEGRATION: USE OF PARTIAL FRACTIONS AND ALGEBRAIC LONG DIVISION**

**1.** Make use of partial fractions to find:

(a) $\int \frac{3x}{(x-1)(x-2)}\,dx$ (b) $\int \frac{x-3}{x(x-1)}\,dx$ (c) $\int \frac{5x+1}{(x-1)(x+2)}\,dx$

(d) $\int \frac{5-x}{(x-1)(x+3)}\,dx$ (e) $\int \frac{3x+5}{(x+1)(x+3)}\,dx$ (f) $\int \frac{x-8}{(x+1)(2x-1)}\,dx$

(g) $\int \frac{2-3x}{x(1-x)}\,dx$ (h) $\int \frac{1}{(2x+3)(x+2)}\,dx$ (i) $\int \frac{2}{x^2-1}\,dx$

(j) $\int \frac{3x-5}{x^2-3x+2}\,dx$ (k) $\int \frac{x+6}{x^2+2x}\,dx$ (l) $\int \frac{10}{2x^2+3x-2}\,dx$

(m) $\int \frac{x-15}{2x^2-5x-12}\,dx$ (n) $\int \frac{2(x+1)}{x^2+2x-3}\,dx$ (o) $\int \frac{6x+5}{3x^2+5x-2}\,dx$

(p) $\int \frac{x+7}{x^2+8x+15}\,dx$ (q) $\int \frac{x}{x^2-2x-3}\,dx$ (r) $\int \frac{x+1}{2x^2+x}\,dx$

**2.** Evaluate, giving each answer correct to 3 decimal places:

(a) $\int_4^6 \frac{4x-9}{(x-2)(x-3)}\,dx$ (b) $\int_5^6 \frac{1}{(x-2)(x-4)}\,dx$ (c) $\int_3^4 \frac{5}{(x+3)(x-2)}\,dx$

(d) $\int_1^2 \frac{x+2}{x(x+4)}\,dx$ (e) $\int_2^3 \frac{1}{(4-x)(x-1)}\,dx$ (f) $\int_8^{12} \frac{6(x-5)}{(x-4)(x-7)}\,dx$

(g) $\int_4^5 \frac{2x}{x^2-4x+3}\,dx$ (h) $\int_5^6 \frac{x+24}{x^2-x-12}\,dx$ (i) $\int_2^3 \frac{x-2}{2x^2-x-3}\,dx$

(j) $\int_0^4 \frac{3x+7}{x^2+5x+6}\,dx$ (k) $\int_2^6 \frac{x-11}{2x^2+x-3}\,dx$ (l) $\int_2^4 \frac{3}{2x^2+5x+2}\,dx$

**3.** Using algebraic long division first, find:

(a) $\int \frac{x}{x+2}\,dx$ (b) $\int \frac{x}{x-1}\,dx$ (c) $\int \frac{x}{x-4}\,dx$

(d) $\int \frac{x}{x+6}\,dx$ (e) $\int \frac{x+5}{x+2}\,dx$ (f) $\int \frac{x+1}{x+3}\,dx$

(g) $\int \frac{2x+3}{2x+1}\,dx$ (h) $\int \frac{x^2+4x+2}{x+1}\,dx$ (i) $\int \frac{x^2-x+3}{x-1}\,dx$

(j) $\int \frac{x^2+6x+7}{x+4}\,dx$ (k) $\int \frac{2x^2-3x+2}{x-2}\,dx$ (l) $\int \frac{3x^2-2x+1}{3x+1}\,dx$

(m) $\int \frac{x^3+5x^2+5x-1}{x+3}\,dx$ (n) $\int \frac{2x^3-7x^2+7x-1}{2x-1}\,dx$

(o) $\int \frac{2x^3+3x^2-2x-5}{x+1}\,dx$ (p) $\int \frac{x^3-4x^2-3x+14}{x-4}\,dx$

(q) $\int \frac{x-3}{x-1}\,dx$ (r) $\int \frac{3x+2}{x-2}\,dx$ (s) $\int \frac{2x}{x+3}\,dx$

(t) $\int \frac{3x+1}{2x+1}\,dx$ (u) $\int \frac{4x}{x-1}\,dx$ (v) $\int \frac{2x^2+7x+1}{x+2}\,dx$

(w) $\int \frac{x+2}{x-2}\,dx$ (x) $\int \frac{x^2}{x+3}\,dx$ (y) $\int \frac{x(2x+1)}{x+1}\,dx$

(z) $\int \frac{x^2+4}{x+1}\,dx$

**4.** Evaluate, giving each answer correct to 3 decimal places:

(a) $\int_5^7 \frac{x}{x+5}\,dx$ (b) $\int_3^5 \frac{x}{x-2}\,dx$ (c) $\int_1^3 \frac{x+2}{x+1}\,dx$

(d) $\int_3^8 \frac{x^2+7x+6}{x+2}\,dx$ (e) $\int_0^4 \frac{x(x+3)}{x+2}\,dx$ (f) $\int_1^3 \frac{x^2-2x-2}{x+1}\,dx$

(g) $\int_1^3 \frac{2x^2+5x-1}{x+3}\,dx$ (h) $\int_1^4 \frac{2x^2+5x+1}{2x+1}\,dx$ (i) $\int_0^1 \frac{2x^2}{2x+1}\,dx$

(j) $\int_{4}^{10} \frac{x^3 + x^2 - 8x + 8}{x - 2} dx$ (k) $\int_{0}^{1} \frac{8x}{4x+3} dx$ (l) $\int_{0}^{1} \frac{x+1}{2x+1} dx$

**5.** Using algebraic long division first and then partial fractions, find:

(a) $\int \frac{x^2 + 2x + 2}{x^2 + x} dx$ (b) $\int \frac{x^2}{x^2 - 4} dx$ (c) $\int \frac{x^2 + 3x - 13}{x^2 + x - 2} dx$

(d) $\int \frac{x^2 - 4x - 2}{x^2 - 2x} dx$ (e) $\int \frac{2x^2 + 2x + 3}{x^2 - 1} dx$ (f) $\int \frac{x^2 + 2}{x^2 + 2x} dx$

(g) $\int \frac{2x^3 + 4x^2 - 15x - 8}{x^2 + 2x - 8} dx$ (h) $\int \frac{x^2 + 3}{x^2 - 1} dx$

(i) $\int \frac{x^2 + 2x - 6}{x^2 + x - 2} dx$ (j) $\int \frac{x^3 - x^2 - 5x + 1}{x^2 - 2x - 3} dx$

(k) $\int \frac{x^3 - 5x^2 + 11x - 12}{x^2 - 5x + 6} dx$ (l) $\int \frac{x^3 + 4x^2 - x + 2}{x^2 + x} dx$

(m) $\int \frac{x^2 + 2x - 3}{x^2 + 3x + 2} dx$ (n) $\int \frac{x^3 + 2x^2 - 5x - 6}{x^2 - 4x + 3} dx$

(o) $\int \frac{2x^3 + x^2 - 3x + 11}{2x^2 + 3x - 2} dx$

# ANSWERS

**1**
(a) $-3\ln(x-1)+6\ln(x-2)+C$
(b) $3\ln x-2\ln(x-1)+C$
(c) $2\ln(x-1)+3\ln(x+2)+C$
(d) $\ln(x-1)-2\ln(x+3)+C$
(e) $\ln(x+1)+2\ln(x+3)+C$
(f) $3\ln(x+1)-\frac{5}{2}\ln(2x-1)+C$
(g) $2\ln x+\ln(1-x)+C$
(h) $\ln(2x+3)-\ln(x+2)+C$
(i) $\ln(x-1)-\ln(x+1)+C$
(j) $2\ln(x-1)+\ln(x-2)+C$
(k) $3\ln x-2\ln(x+2)+C$
(l) $2\ln(2x-1)-2\ln(x+2)+C$
(m) $\frac{3}{2}\ln(2x+3)-\ln(x-4)+C$
(n) $\ln(x+3)+\ln(x-1)+C$
(o) $\ln(3x-1)+\ln(x+2)+C$
(p) $2\ln(x+3)-\ln(x+5)+C$
(q) $\frac{3}{4}\ln(x-3)+\frac{1}{4}\ln(x+1)+C$
(r) $\ln x-\frac{1}{2}\ln(2x+1)+C$

**2**
(a) 3·989 (b) 0·203 (c) 0·539 (d) 0·438
(e) 0·462 (f) 7·824 (g) 1·792 (h) 2·419
(i) 0·063 (j) 2·793 (k) −1·314 (l) 0·182

**3**
(a) $x-2\ln(x+2)+C$
(b) $x+\ln(x-1)+C$
(c) $x+4\ln(x-4)+C$
(d) $x-6\ln(x+6)+C$
(e) $x+3\ln(x+2)+C$
(f) $x-2\ln(x+3)+C$
(g) $x+\ln(2x+1)+C$
(h) $\frac{1}{2}x^2+3x-\ln(x+1)+C$
(i) $\frac{1}{2}x^2+3\ln(x-1)+C$
(j) $\frac{1}{2}x^2+2x-\ln(x+4)+C$
(k) $x^2+x+4\ln(x-2)+C$
(l) $\frac{1}{2}x^2-x+\frac{2}{3}\ln(3x+1)+C$
(m) $\frac{1}{3}x^3+x^2-x+2\ln(x+3)+C$
(n) $\frac{1}{3}x^3-\frac{3}{2}x^2+2x+\frac{1}{2}\ln(2x-1)+C$
(o) $\frac{2}{3}x^3+\frac{1}{2}x^2-3x-2\ln(x+1)+C$
(p) $\frac{1}{3}x^3-3x+2\ln(x-4)+C$
(q) $x-2\ln(x-1)+C$
(r) $3x+8\ln(x-2)+C$
(s) $2x-6\ln(x+3)+C$
(t) $\frac{3}{2}x-\frac{1}{4}\ln(2x+1)+C$

(u) $4x + 4\ln(x-1) + C$

(v) $x^2 + 3x - 5\ln(x+2) + C$

(w) $x + 4\ln(x-2) + C$

(x) $\frac{1}{2}x^2 - 3x + 9\ln(x+3) + C$

(y) $x^2 - x + \ln(x+1) + C$

(z) $\frac{1}{2}x^2 - x + 5\ln(x+1) + C$

4. (a) 1.088 (b) 4.197 (c) 2.693 (d) 49.727

(e) 9.803 (f) −1.307 (g) 6.811 (h) 12.951

(i) 0.275 (j) 431.545 (k) 0.729 (l) 0.775

5. (a) $x + 2\ln x - \ln(x+1) + C$

(b) $x + \ln(x-2) - \ln(x+2) + C$

(c) $x + 5\ln(x+2) - 3\ln(x-1) + C$

(d) $x + \ln x - 3\ln(x-2) + C$

(e) $2x + \frac{7}{2}\ln(x-1) - \frac{3}{2}\ln(x+1) + C$

(f) $x + \ln x - 3\ln(x+2) + C$

(g) $x^2 + 2\ln(x+4) - \ln(x-2) + C$

(h) $x + 2\ln(x-1) - 2\ln(x+1) + C$

(i) $x + 2\ln(x+2) - \ln(x-1) + C$

(j) $\frac{1}{2}x^2 + x + \ln(x-3) - \ln(x+1) + C$

(k) $\frac{1}{2}x^2 + 3\ln(x-3) + 2\ln(x-2) + C$

(l) $\frac{1}{2}x^2 + 3x + 2\ln x - 6\ln(x+1) + C$

(m) $x + 3\ln(x+2) - 4\ln(x+1) + C$

(n) $\frac{1}{2}x^2 + 6x + 12\ln(x-3) + 4\ln(x-1) + C$

(o) $\frac{1}{2}x^2 - x + 2\ln(2x-1) - \ln(x+2) + C$

ADVANCED HIGHER MATHEMATICS

## INTEGRATION BY PARTS

1. Use the method of integration by parts to find each of these indefinite integrals.

(a) $\int x\cos x\,dx$ (b) $\int x\sin x\,dx$ (c) $\int x\sin 2x\,dx$

(d) $\int x\cos 3x\,dx$ (e) $\int x\sin 4x\,dx$ (f) $\int x\sin\left(\frac{1}{2}x\right)dx$

2. Use the method of integration by parts to find each of these indefinite integrals.

(a) $\int xe^{x}dx$ (b) $\int xe^{2x}dx$ (c) $\int xe^{3x}dx$

(d) $\int xe^{-x}dx$ (e) $\int xe^{-4x}dx$ (f) $\int (x+1)e^{4x}dx$

(g) $\int (2x+3)e^{5x}dx$ (h) $\int (1-3x)e^{-2x}dx$

3. Use the method of integration by parts to find each of these indefinite integrals.

(a) $\int x\ln x\,dx$ (b) $\int x^2\ln x\,dx$ (c) $\int x^4\ln x\,dx$

(d) $\int \frac{1}{x^2}\ln x\,dx$ (e) $\int \frac{1}{x^3}\ln x\,dx$

4. Use the method of integration by parts to find each of these indefinite integrals.

(a) $\int x(x+1)^3dx$ (b) $\int x(x-1)^4dx$ (c) $\int x(x+1)^5dx$

(d) $\int x(2x+1)^3dx$ (e) $\int x(3x+1)^4dx$ (f) $\int (4x+1)(2x+3)^4dx$

(g) $\int (2x+3)(2x-1)^5dx$ (f) $(3x+1)(x+1)^8dx$

5. Use the method of integration by parts to find each of these indefinite integrals.

(a) $\int x\sqrt{x+4}\,dx$ (b) $\int x\sqrt{x-3}\,dx$ (c) $\int \frac{x}{\sqrt{x+1}}dx$

(d) $\int x\sqrt{2x+1}\,dx$ (e) $\int \frac{x}{\sqrt{x-2}}dx$ (f) $\int x\sqrt{3x+2}\,dx$

(g) $\int x(2x+1)^{\frac{1}{3}}dx$ (h) $\int \frac{2x+5}{\sqrt{3x-1}}dx$

6. $\int \ln x\,dx$ can also be found using the method of integration by parts by noting that $\int \ln x\,dx = \int (\ln x)\cdot 1dx$.

Now let $u=\ln x$ and $\frac{dv}{dx}=1$, and complete the integration.

7. (a) Given $y=\ln(1+x^2)$, find $\frac{dy}{dx}$.

(b) $\int \tan^{-1}x\,dx$ can be found using the method of integration by parts by noting that $\int \tan^{-1}x\,dx = \int (\tan^{-1}x)\cdot 1dx$.

Now let $u=\tan^{-1}x$ and $\frac{dv}{dx}=1$, and complete the integration (you will need to make use of your answer to part (a)).

8. (a) Given $y=xe^x$, find $\frac{dy}{dx}$ in its fully factorised form.

(b) Hence use the method of integration by parts to find $\int \frac{xe^x}{(x+1)^2}dx$.

9. Use successive integration by parts to find each of these indefinite integrals.

(a) $\int x^2e^xdx$ (b) $\int x^2\sin x\,dx$ (c) $\int x^2\cos x\,dx$

(d) $\int x^2e^{-x}dx$ (e) $\int x^2e^{2x}dx$ (f) $\int x^2e^{3x}dx$

(g) $\int x^2\cos 2x\,dx$ (h) $\int x^2\sin 4x\,dx$ (i) $\int x^2(x+1)^5dx$

(j) $\int x^2\sqrt{x+1}\,dx$ (k) $\int x^2(2x+1)^4dx$ (l) $\int x^2\sqrt{2x+1}\,dx$

(m) $\int x^2\cos 3x\,dx$ (n) $\int x^2\sin 2x\,dx$

10. Use successive integration by parts to find $\int x^3e^xdx$.

11. Use the method of integration by parts to evaluate each of these definite integrals.

(a) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (b) $\int_0^{\pi} x\sin x\,dx$ (c) $\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$

(d) $\int_0^{\pi} x\sin\left(\frac{1}{2}x\right)dx$ (e) $\int_0^{1} xe^{x}\,dx$ (f) $\int_0^{1} xe^{5x}\,dx$

(g) $\int_0^{1} xe^{-2x}\,dx$ (h) $\int_0^{1} x(x+1)^4\,dx$ (i) $\int_0^{3} x\sqrt{x+1}\,dx$

(j) $\int_3^{6} \frac{x}{\sqrt{x-2}}\,dx$

12. Use the method of integration by parts to evaluate each of these definite integrals correct to 3 decimal places.

(a) $\int_1^{2} x^3 \ln x\,dx$ (b) $\int_1^{4} \frac{1}{x^3}\ln x\,dx$

13. Study the worked example below carefully.

**Worked Example**

Use the method of integration by parts to find $\int e^x \cos x\,dx$.

**Solution**

Let $I = \int e^x \cos x\,dx$.

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$.

Then $\frac{du}{dx} = e^x$ and $v = \sin x$.

$I = uv - \int v\frac{du}{dx}dx$

$= e^x \sin x - \int e^x \sin x\,dx$

$= e^x \sin x - J$, where $J = \int e^x \sin x\,dx$.

Now let $u = e^x$ and $\frac{dv}{dx} = \sin x$.

Then $\frac{du}{dx} = e^x$ and $v = -\cos x$.

$J = uv - \int v\frac{du}{dx}dx$

$= -e^x \cos x - \int -e^x \cos x\,dx$

$= -e^x \cos x + \int e^x \cos x\,dx$

$= -e^x \cos x + I$

Now $I = e^x \sin x - J$

$= e^x \sin x + e^x \cos x - I$.

Hence $2I = e^x \sin x + e^x \cos x$,

and so $I = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$

$= \frac{1}{2}e^x(\sin x + \cos x) + C$.

Now use the method of integration by parts to find:

(a) $\int e^{2x} \cos x dx$  (b) $\int e^{x} \cos 2x dx$

(c) Also, use the method of integration by parts to evaluate $\int_0^{\pi} e^{x} \sin x dx$,

expressing your answer in terms of $e$ and $\pi$.

## ANSWERS

1. (a) $x \sin x + \cos x + C$  (b) $-x \cos x + \sin x + C$
   (c) $-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$  (d) $\frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C$
   (e) $-\frac{1}{4} x \cos 4x + \frac{1}{16} \sin 4x + C$  (f) $-2x \cos\left(\frac{1}{2}x\right) + 4 \sin\left(\frac{1}{2}x\right) + C$

2. (a) $xe^{x} - e^{x} + C$  (b) $\frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + C$
   (c) $\frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} + C$  (d) $-xe^{-x} - e^{-x} + C$
   (e) $-\frac{1}{4} xe^{-4x} - \frac{1}{16} e^{-4x} + C$  (f) $\frac{1}{4}(x+1)e^{4x} - \frac{1}{16} e^{4x} + C$
   (g) $\frac{1}{5}(2x+3)e^{5x} - \frac{2}{25} e^{5x} + C$  (h) $-\frac{1}{2}(1-3x)e^{-2x} + \frac{3}{4} e^{-2x} + C$

3. (a) $\frac{1}{2} x^{2} \ln x - \frac{1}{4} x^{2} + C$  (b) $\frac{1}{3} x^{3} \ln x - \frac{1}{9} x^{3} + C$
   (c) $\frac{1}{5} x^{5} \ln x - \frac{1}{25} x^{5} + C$  (d) $-\frac{1}{x} \ln x - \frac{1}{x} + C$
   (e) $-\frac{1}{2x^{2}} \ln x - \frac{1}{4x^{2}} + C$

4. (a) $\frac{1}{4} x(x+1)^{4} - \frac{1}{20}(x+1)^{5} + C$  (b) $\frac{1}{5} x(x-1)^{5} - \frac{1}{30}(x-1)^{6} + C$
   (c) $\frac{1}{6} x(x+1)^{6} - \frac{1}{42}(x+1)^{7} + C$  (d) $\frac{1}{8} x(2x+1)^{4} - \frac{1}{80}(2x+1)^{5} + C$
   (e) $\frac{1}{5} x(3x+1)^{5} - \frac{1}{270}(3x+1)^{6} + C$
   (f) $\frac{1}{10}(4x+1)(2x+3)^{5} - \frac{1}{30}(2x+3)^{6} + C$
   (g) $\frac{1}{12}(2x+3)(2x-1)^{6} - \frac{1}{84}(2x-1)^{7} + C$
   (h) $\frac{1}{9}(3x+1)(x+1)^{9} - \frac{1}{30}(x+1)^{10} + C$

5. (a) $\frac{2}{3}x(x+4)^{\frac{3}{2}} - \frac{4}{15}(x+4)^{\frac{5}{2}} + C$ (b) $\frac{2}{3}x(x-3)^{\frac{3}{2}} - \frac{4}{15}(x-3)^{\frac{5}{2}} + C$

(c) $2x\sqrt{x+1} - \frac{4}{3}(x+1)^{\frac{3}{2}} + C$ (d) $\frac{1}{3}x(2x+1)^{\frac{3}{2}} - \frac{1}{15}(2x+1)^{\frac{5}{2}} + C$

(e) $2x\sqrt{x-2} - \frac{4}{3}(x-2)^{\frac{3}{2}} + C$ (f) $\frac{2}{9}x(3x+2)^{\frac{3}{2}} - \frac{4}{135}(3x+2)^{\frac{5}{2}} + C$

(g) $\frac{3}{8}x(2x+1)^{\frac{4}{3}} - \frac{9}{112}(2x+1)^{\frac{7}{3}} + C$ (h) $\frac{2}{3}(2x+5)\sqrt{3x-1} - \frac{8}{27}(3x-1)^{\frac{3}{2}} + C$

6. $\int \ln x\,dx = x\ln x - x + C$

7. (a) $\frac{dy}{dx} = \frac{2x}{1+x^2}$ (b) $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$

8. (a) $\frac{dy}{dx} = (x+1)e^x$ (b) $\int \frac{xe^x}{(x+1)^2}dx = -\frac{xe^x}{x+1} + e^x + C$

9. (a) $x^2e^x - 2xe^x + 2e^x + C$ (b) $-x^2\cos x + 2x\sin x + 2\cos x + C$

(c) $x^2\sin x + 2x\cos x - 2\sin x + C$ (d) $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

(e) $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ (f) $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

(g) $\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

(h) $-\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C$

(i) $\frac{1}{6}x^2(x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{168}(x+1)^8 + C$

(j) $\frac{2}{3}x^2(x+1)^{\frac{3}{2}} - \frac{8}{15}x(x+1)^{\frac{5}{2}} + \frac{16}{105}(x+1)^{\frac{7}{2}} + C$

(k) $\frac{1}{10}x^2(2x+1)^5 - \frac{1}{60}x(2x+1)^6 + \frac{1}{840}(2x+1)^7 + C$

(l) $\frac{1}{3}x^2(2x+1)^{\frac{3}{2}} - \frac{2}{15}x(2x+1)^{\frac{5}{2}} + \frac{2}{105}(2x+1)^{\frac{7}{2}} + C$

(m) $\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$

(n) $-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

10. $\int x^3e^x dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

11. (a) $\frac{\pi}{2} - 1$ (b) $\pi$ (c $\frac{\pi}{8} - \frac{1}{4}$ (d) 4 (e) 1

(f) $\frac{1}{25}(4e^5 + 1)$ (g) $\frac{1}{4}(1 - 3e^{-2})$ (h) $4\frac{3}{10}$ (i) $7\frac{11}{15}$ (j) $8\frac{2}{3}$

12. (a) 1.835 (b) 0.191

13. (a) $\frac{1}{5}e^{2x}(\sin x + 2\cos x) + C$ (b) $\frac{1}{5}e^x(2\sin 2x + \cos 2x) + C$

(c) $\frac{1}{2}(e^\pi + 1)$

ADVANCED HIGHER MATHEMATICS

## HOMEWORK ON INTEGRATION BY PARTS

**1.** Use the method of integration by parts to find each indefinite integral.

(a) $\int x\cos 4x\,dx$ (b) $\int xe^{2x}\,dx$ (c) $\int x\ln x\,dx$

(d) $\int x(2x+1)^5\,dx$ (e) $\int x\sqrt{x+1}\,dx$ (f) $\int \ln x\,dx$

**2.** Use the method of integration by parts to find $\int \cos^{-1} x\,dx$.

**3.** Use successive integration by parts to find each indefinite integral.

(a) $\int x^2e^x\,dx$ (b) $\int x^2\sin x\,dx$ (c) $\int x^2(x+1)^4\,dx$

**4.** Use the method of integration by parts to evaluate each definite integral.

(a) $\int_0^{\pi} x\sin x\,dx$ (b) $\int_0^1 xe^{-x}\,dx$ (c) $\int_0^4 x\sqrt{2x+1}\,dx$

**5.** Use the method of integration by parts to evaluate the definite integral $\int_1^2 x^3\ln x\,dx$, giving your answer correct to 3 decimal places.

**6** Use integration by parts **twice** to find $\int e^x\sin x\,dx$.

**7** Let $Q_n = \int_0^{\pi} x^n\sin x\,dx$, where $n \geq 0$.

(a) Use integration by parts to show that

$$Q_n = \pi^n - n(n-1)Q_{n-2}, \qquad \text{where } n \geq 1.$$

(b) Evaluate $Q_0$.

(c) Using your answers to parts (a) and (b), hence evaluate the definite integral $\int_0^{\pi} x^4\sin x\,dx$.

## VERIFICATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

1. Verify that the function $y = x^4$ satisfies the differential equation
$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0.$$

2. Verify that the function $y = Ae^{3x} + Be^{x}$, where $A$ and $B$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0.$$

3. Verify that the function $y = Ae^{x} + Be^{2x}$, where $A$ and $B$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

4. (a) Verify that the function $y = A\sin 2x + B\cos 2x$, where $A$ and B are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} + 4y = 0.$$

   (b) Verify that, in general, the function $y = A\sin kx + B\cos kx$, where $A$, $B$ and $k$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} + k^2y = 0.$$

5. (a) Verify that the function $y = Ae^{3x} + Be^{-3x}$, where $A$ and $B$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} - 9y = 0.$$

   (b) Verify that, in general, the function $y = Ae^{kx} + Be^{-kx}$, where $A$, $B$ and $k$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} - k^2y = 0.$$

6. Verify that the function $y = x^n$, where $n$ is a constant, satisfies the differential equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0.$$

7. Verify that the function $y = Ae^{5x} + Be^{-3x} - \frac{1}{7}e^{4x}$, where $A$ and $B$ are constants, satisfies the differential equation
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = e^{4x}.$$

8. Verify that the function $y = e^{x}\sin x$ satisfies the differential equation
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

9. Verify that the function $y = x^3\ln x$ satisfies the differential equation
$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0.$$

10. Verify that the function $y = e^{-x}\cos 2x$ satisfies the differential equation
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

## DIFFERENTIAL EQUATIONS 1

1. Find the general solution of each differential equation below.

(a) $\frac{dy}{dx} = \frac{x}{2y}$ (b) $\frac{dy}{dx} = \frac{x}{y^2}$

(c) $\frac{dy}{dx} = \frac{x^3}{y}$ (d) $\frac{dy}{dx} = \frac{x+2}{y}$

(e) $2y\frac{dy}{dx} = 3x^2$ (f) $y^2\frac{dy}{dx} = x^5$

(g) $\frac{dy}{dx} = \frac{2}{\cos y}$ (h) $\frac{dy}{dx} = \frac{\cos x}{y}$

(i) $\sin y\frac{dy}{dx} = 1$ (j) $\cos 2y\frac{dy}{dx} - x = 0$

(k) $\frac{dy}{dx} = \frac{4x}{e^y}$ (l) $\frac{dy}{dx} = \frac{e^x}{y}$

(m) $e^y\frac{dy}{dx} = 1$ (n) $\frac{dy}{dx} = \frac{x}{e^{2y}}$

(o) $3y\frac{dy}{dx} = 5x^2$ (p) $x + 4y\frac{dy}{dx} = 0$

(q) $y^2\frac{dy}{dx} = x + 1$ (r) $\frac{dy}{dx} = \frac{\sin x}{\cos y}$

(s) $\frac{dy}{dx} = \frac{x^2 - 1}{y^3}$ (t) $x^2 + 3y\frac{dy}{dx} = 0$

(u) $\frac{dy}{dx} = 4x\sqrt{y}$ (v) $e^{4y}\frac{dy}{dx} = x^2(x+1)$

(w) $y\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ (x) $1 + y\frac{dy}{dx} = x$

(y) $y\frac{dy}{dx} = \sqrt{x}$ (z) $\sin y\frac{dy}{dx} = e^{2x}$

2. Find the general solution of each differential equation below.

(a) $\frac{dy}{dx} = 4x\sqrt{1-y^2}$ (b) $\frac{dy}{dx} = 2x(1+y^2)$

(c) $\frac{dy}{dx} = 2x\sqrt{4-y^2}$ (d) $y\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(e) $(1+x^2)y^2\frac{dy}{dx} = 1$ (f) $\frac{dy}{dx} = x(4+y^2)$

3. (a) Use integration by parts to find $\int xe^x dx$.

(b) Hence find the general solution of the differential equation $\frac{dy}{dx} = \frac{xe^x}{y}$.

4. (a) Use integration by parts to find $\int x\sin x dx$.

(b) Hence find the general solution of the differential equation $\frac{dy}{dx} = \frac{x\sin x}{e^y}$.

5. Study the worked example below carefully.

**Worked Example**

Find the general solution of the differential equation

$$\frac{dy}{dx} = y^2 \sin x.$$

**Solution**

$$\frac{dy}{dx} = y^2\sin x \quad \Rightarrow \quad dy = y^2 \sin x dx$$
$$\Rightarrow \quad \int \frac{1}{y^2} dy = \int \sin x dx$$
$$\Rightarrow \quad \int y^{-2} dy = \int \sin x dx$$
$$\Rightarrow \quad \frac{1}{(-1)} y^{-1} = -\cos x + C$$
$$\Rightarrow \quad -\frac{1}{y} = -\cos x + C \qquad \{\times(-1)\}$$
$$\Rightarrow \quad \frac{1}{y} = \cos x + C$$
$$\Rightarrow \quad y = \frac{1}{\cos x + C}$$

(a) Show that the general solution of the differential equation

$$\frac{dy}{dx} = e^x y^2$$

can be expressed in the form $y = \frac{1}{C - e^x}$, where $C$ is a constant.

(b) Show that the general solution of the differential equation

$$x^5 \frac{dy}{dx} = 4y^2$$

can be expressed in the form $y = \frac{x^4}{1 + Cx^4}$, where $C$ is a constant.

## ANSWERS

1. (a) $2y^2 = x^2 + C$ (b) $2y^3 = 3x^2 + C$ (c) $2y^2 = x^4 + C$

(d) $y^2 = x^2 + 4x + C$ (e) $y^2 = x^3 + C$ (f) $2y^3 = x^6 + C$

(g) $\sin y = 2x + C$ (h) $y^2 = 2\sin x + C$ (i) $\cos y = C - x$

(j) $\sin 2y = x^2 + C$ (k) $y = \ln(2x^2 + C)$ (l) $y^2 = 2e^x + C$

(m) $y = \ln(x + C)$ (n) $y = \frac{1}{2}\ln(x^2 + C)$ (o) $9y^2 = 10x^3 + C$

(p) $4y^2 = C - x^2$ (q) $2y^3 = 3x^2 + 6x + C$ (r) $\sin y = C - \cos x$

(s) $3y^4 = 4x^3 - 12x + C$ (t) $9y^2 = C - 2x^3$ (u) $y = (x^2 + C)^2$

(v) $y = \frac{1}{4}\ln(x^4 + 2x^2 + C)$ (w) $y^2 = 4\sqrt{x} + C$ (x) $y^2 = x^2 - 2x + C$

(y) $3y^2 = 4x^{\frac{3}{2}} + C$ (z) $2\cos y = C - e^{2x}$

2. (a) $y = \sin(2x^2 + C)$ (b) $y = \tan(x^2 + C)$ (c) $y = 2\sin(x^2 + C)$

(d) $y^2 = 2\sin^{-1} x + C$ (e) $y^3 = 3\tan^{-1} x + C$ (f) $y = 2\tan(x^2 + C)$

3. (a) $\int xe^x dx = xe^x - e^x + C$ (b) $y^2 = 2(x-1)e^x + C$

4. (a) $\int x \sin x dx = -x\cos x + \sin x + C$ (b) $y = \ln(\sin x - x\cos x + C)$