## DIFFERENTIAL EQUATIONS: HOMEWORK 1

1. Find the particular solution of each differential equation below, in each case expressing $y$ explicitly in terms of $x$.

(a) $\frac{dy}{dx} = \frac{x(x-2)}{y^2}$; $\quad y = 1$ when $x = 2$

(b) $e^{2y}\frac{dy}{dx} = x + 1$; $\quad y = 0$ when $x = 1$

(c) $\sqrt{x}\frac{dy}{dx} - y = 0$; $\quad y = 3$ when $x = 0$

(d) $\frac{dy}{dx} = \frac{y-1}{x+3}$; $\quad y = 3$ when $x = -2$

(e) $(2x+1)\frac{dy}{dx} + y = 0$; $\quad y = 2$ when $x = 4$

2. (a) Use integration by parts to find $\int x\cos x\,dx$.

(b) Hence find the general solution of the differential equation

$$e^y\frac{dy}{dx} - x\cos x = 0.$$

(c) Find the particular solution of the differential equation in (b) if $y = 0$ when $x = \pi$, expressing $y$ explicitly in terms of $x$.

3. (a) Express $\frac{2}{x(x+2)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx} = \frac{2y}{x(x+2)},$$

given that $y = 2$ when $x = 4$, expressing $y$ explicitly in terms of $x$.

4. (a) Express $\frac{3(x-2)}{x(x-3)}$ in partial fractions.

(b) Hence solve the differential equation

$$x(x-3)\frac{dy}{dx} = 3y(x-2),$$

given that $y = 8$ when $x = 4$, expressing $y$ explicitly in terms of $x$.

5. (a) Express $\frac{1}{y(y+1)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx} = 2y(y+1),$$

given that $y = 3$ when $x = 0$, expressing $y$ explicitly in terms of $x$.

6. (a) Express $\frac{1}{(y+1)(y+2)}$ in partial fractions.

(b) Hence show that the solution of the differential equation

$$\frac{dy}{dx} = \frac{3(y+1)(y+2)}{x}$$

can be expressed in the form $y = \frac{2Ax^3 - 1}{1 - Ax^3}$, where $A$ is a constant.

(c) Find the particular solution of the differential equation in (b) if $y = -3$ when $x = 1$.

ADVANCED HIGHER MATHEMATICS

## EXAMINATION QUESTIONS ON DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

**1.** A mathematical biologist believes that the differential equation

$$y\frac{dy}{dx} - 3x = x^4$$

models a process.

Given that $y = 2$ when $x = 1$, obtain $y$ in terms of $x$.

**2.** (a) A function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

Given that $y = 2$ when $x = 1$, obtain $y$ as a function of $x$.

(b) A function $x(t)$ satisfies the differential equation $\frac{dx}{dt} = -2x^3$.

Given that $x = 1$ when $t = 0$, obtain $x$ as a function of $t$ for $t \geq 0$

**3.** The number of strands of bacteria, $B(t)$, present in a culture after $t$ days of growth is assumed to be increasing at a rate proportional to the number of bacteria present.

(a) Write down a differential equation for $B$ and find the general solution for $B$ in terms of $t$.

(b) Given that the number of strands observed after 1 day is 502 and after 4 days is 1833, find the number of strands initially present.

**4.** A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount $M$ grams of plant food effective after $t$ days satisfies the differential equation

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant.}$$

(a) Find the general solution for $M$ in terms of $t$ where the initial amount of plant food is 100 grams.

(b) Find the value of $k$ if after 30 days only half the initial amount of plant food is effective.

(c) What percentage of the original amount of plant food is effective after 35 days?

(d) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer of the plant food justified in calling its product "sixty day super food"?

**5.** When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water.

This can be represented by the differential equation

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}, \quad h \geq 0,$$

where $h$ is the depth (in metres) of the water and $t$ is the time (in minutes) elapsed since the valve was opened.

(a) Express $h$ as a function of $t$ given that the pool was initially 9 metres deep.

(b) How long did it take to drain the pool?

**6.** The volume $V(t)$ of a cell at time $t$ changes according to the law

$$\frac{dV}{dt} = V(10 - V) \qquad \text{for } 0 < V < 10.$$

(a) Show that

$$\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$$

for some constant $C$.

(b) Given that $V(0) = 5$, show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$

(c) Obtain the limiting value of $V(t)$ as $t \to \infty$

## DIFFERENTIAL EQUATIONS: HOMEWORK 2

1. The number of strands of bacteria, $N$, present in a culture after $t$ days of growth is assumed to be increasing at a rate proportional to the number of strands present.

   (a) Write down a differential equation satisfied by $N$ and hence express $N$ as a function of $t$.

   (b) Given that there are 200 strands initially present, and that the number of strands observed after one week is 3850, estimate the number of strands likely to be present after two weeks, to the nearest thousand.

2. An outdoor thermometer registering a temperature of $40°$ is brought into a room where the temperature is $70°$. The temperature registered on the thermometer will gradually increase from $40°$ until the thermometer eventually registers the room temperature of $70°$.

   Let $T°$ be the temperature registered on the thermometer $t$ minutes after it is brought into the room. The rate at which $T$ varies is represented by the differential equation

$$\frac{dT}{dt} = k(70 - T),$$

   where $k$ is a constant.

   Given that the thermometer registers $60°$ after 5 minutes, after how many minutes will the thermometer register a temperature of $65°$?

3. A scientist grows a culture of bacteria in his laboratory. The number of bacteria, $B$, after $t$ hours is assumed to satisfy the differential equation

$$\frac{dB}{dt} = kB,$$

   where $k$ is a constant.

   Records indicate that there are 650 bacteria after 2 hours, and 900 bacteria after 5 hours.

   (a) Find the initial number of bacteria in the culture.

   (b) Estimate the number of bacteria at the end of the first day of growth.

4. When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water. Let $h$ be the depth (in metres) of the water $t$ minutes after the valve was opened.

   (a) Write down a differential equation satisfied by $h$ and find the general solution of your differential equation.
   (You need **not** express $h$ explicitly as a function of $t$.)

   (b) Given that the pool was initially 9 metres deep, and that the depth of water was $2\cdot25$ metres after half an hour of draining:
   (i) find the depth of water after 12 minutes of draining
   (ii) how long will it take for for the pool to drain to a depth of 1 metre?

5. (a) Express $\dfrac{1}{x(1-x)}$ in partial fractions.

   (b) The spread of a disease in a large population can be modelled by means of a differential equation. The **proportion** $x$ of the population infected with the disease after $t$ days satisfies a differential equation of the form

$$\frac{dx}{dt} = kx(1-x), \qquad \text{where } k \text{ is a constant.}$$

   (i) Find the general solution of the differential equation, expressing $x$ explicitly as a function of $t$.

   (ii) Given that $x = \frac{1}{250}$ when $t = 0$ and $x = \frac{1}{100}$ when $t = 3$, verify that about 8% of the population was infected after ten days, and find after how many days 20% of the population will become infected.

6. In a chemical reaction, two substances $X$ and $Y$ combine to form a third substance $Z$. Let $Q$ denote the number of grams of $Z$ formed $t$ minutes after the reaction begins. The rate at which $Q$ varies is represented by the differential equation

$$\frac{dQ}{dt} = \frac{(40-Q)(20-Q)}{600}.$$

   (a) Express $\dfrac{600}{(40-Q)(20-Q)}$ in partial fractions.

   (b) Use your answer to (a) to find the general solution of the differential equation, expressing $Q$ explicitly as a function of $t$.

   Given that $Q = 0$ when $t = 0$, find, correct to one decimal place:
   (i) the number of grams of $Z$ formed one hour after the reaction begins
   (ii) the time taken to form 10 grams of $Z$.