X100/701

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.

2. There are five Sections in this paper.

    Section A assesses the compulsory units Mathematics 1 and 2
    Section B assesses the optional unit Mathematics 3
    Section C assesses the optional unit Statistics 1
    Section D assesses the optional unit Numerical Analysis 1
    Section E assesses the optional unit Mechanics 1.

    Candidates must attempt Section A (Mathematics 1 and 2) **and one** of the following Sections:

    Section B (Mathematics 3)
    Section C (Statistics 1)
    Section D (Numerical Analysis 1)
    Section E (Mechanics 1).

3. **Candidates must use a separate answer book for each Section.** Take care to show clearly the optional section chosen. On the front of the answer book, in the top right hand corner, write B, C, D or E.

4. A booklet of Mathematical Formulae and Statistical Tables is supplied for all candidates. It contains Numerical Analysis formulae and Statistical formulae and tables.

5. **Full credit will be given only where the solution contains appropriate working.**

A1. Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y + 3z &= 2 \\ 2x + y + z &= 2 \\ 3x + 2y + 5z &= 5. \end{aligned}$$

5

A2. Verify that i is a solution of $z^4 + 4z^3 + 3z^2 + 4z + 2 = 0$.
Hence find all the solutions.

5

A3. A curve is defined by the parametric equations

$$x = t^2 + t - 1, \ \ y = 2t^2 - t + 2$$

for all t. Show that the point A (−1, 5) lies on the curve and obtain an equation
of the tangent to the curve at the point A.

6

A4. (a) Given that $f(x) = \sqrt{x}\, e^{-x}$, $x \geq 0$, obtain and simplify $f'(x)$.

4

   (b) Given $y = (x + 1)^2 (x + 2)^{-4}$ and $x > 0$, use logarithmic differentiation to

   show that $\dfrac{dy}{dx}$ can be expressed in the form $\left(\dfrac{a}{x+1} + \dfrac{b}{x+2}\right)y$,

   stating the values of the constants a and b.

3

A5. Use integration by parts to evaluate $\displaystyle\int_0^1 \ln(1+x)\, dx$.

5

A6. Use the substitution $x + 2 = 2 \tan\theta$ to obtain $\displaystyle\int \frac{1}{x^2 + 4x + 8}\, dx$.

5

A7. Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

5

A8. Express $\dfrac{x^2}{(x+1)^2}$ in the form $A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$, $(x \neq -1)$, stating the values of

the constants A, B and C. **3**

A curve is defined by $y = \dfrac{x^2}{(x+1)^2}$, $(x \neq -1)$.

   (i)   Write down equations for its asymptotes. **2**

   (ii) Find the stationary point and justify its nature. **4**

   (iii) Sketch the curve showing clearly the features found in (i) and (ii). **2**

A9. Functions $x(t)$ and $y(t)$ satisfy

$$\frac{dx}{dt} = -x^2 y, \qquad \frac{dy}{dt} = -xy^2.$$

When $t = 0$, $x = 1$ and $y = 2$.

   (a)   Express $\dfrac{dy}{dx}$ in terms of x and y and hence obtain y as a function of x. **5**

   (b)   Deduce that $\dfrac{dx}{dt} = -2x^3$ and obtain x as a function of t for $t \geq 0$. **5**

A10. Define $S_n(x)$ by

$$S_n(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1},$$

where n is a positive integer.

Express $S_n(1)$ in terms of n. **2**

By considering $(1 - x) S_n(x)$, show that

$$S_n(x) = \frac{1 - x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}, \quad x \neq 1.$$ **4**

Obtain the value of $\displaystyle\lim_{n \to \infty} \left\{ \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots + \frac{n}{3^{n-1}} + \frac{3}{2} \cdot \frac{n}{3^n} \right\}.$ **3**

[*END OF SECTION A*]

Candidates should now attempt ONE of the following

**ONLY candidates doing the course Mathematics 1, 2 and 3
should attempt this Section.**

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the
section chosen.

B1.   (*a*)   Find an equation for the plane π_1 which contains the points $A(1, 1, 0)$, $B(3, 1, -1)$ and $C(2, 0, -3)$.   **4**

       (*b*)   Given that π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the planes π_1 and π_2.   **3**

B2.   A matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

     where n is any positive integer.   **6**

B3.   Find the Maclaurin expansion of

$$f(x) = \ln\left(\cos x\right), \; \left(0 \le x < \tfrac{\pi}{2}\right),$$

     as far as the term in x^4.   **5**

B4.   Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of $30°$ about the origin.

     Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\dfrac{kx + y}{2}, \dfrac{x - ky}{2}\right)$, stating the value of k.   **4**

B5.   Find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 5y = 4 \cos x.$$

         6

     Hence determine the solution which satisfies $y(0) = 0$ and $y'(0) = 1$.   **4**

[END OF SECTION B]