Practice Examination C
(Assessing Units 1 & 2)

MATHEMATICS
Advanced Higher Grade

Time allowed - 2 hours

Read Carefully

1. Full credit will be given only where the solution contains appropriate working.
2. **Calculators may be used in this paper.**
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

All questions should be attempted

1. Differentiate with respect to x

   (*a*) $y = \tan^2(3x)$ (3)

   (*b*) $f(x) = \dfrac{x^2}{(3x+1)^6}$ (3)

2. Find the exact value of the recurring decimal

$$0 \cdot 7\dot{6}$$ (4)

3. Prove by induction

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$$ (5)

4. Using partial fractions, prove

$$\frac{1}{1-4x^2} = \frac{1}{2}\left[\frac{1}{2x+1} - \frac{1}{2x-1}\right]$$ (3)

   Hence find the exact value of the integral

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{1-4x^2}\, dx$$ (5)

5. (*a*) Write down the binomial expansion of $(\cos\theta + i\sin\theta)^3$. (3)

   (*b*) Hence use De Moivre's Theorem to find an expression for $\sin^3\theta$ in terms of $\sin 3\theta$ and $\sin\theta$. (4)

6. Find the equation of the tangent to the curve

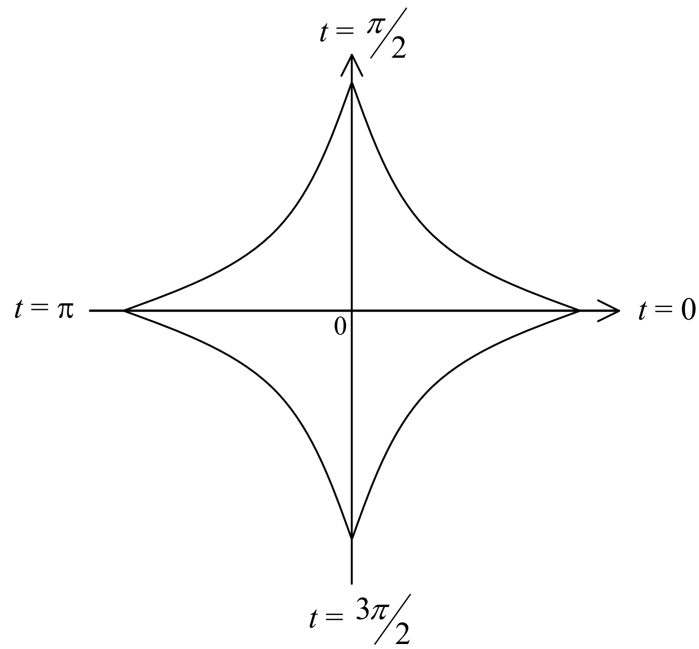
$$x^2 + xy + y^2 = 4$$

   at the point (2, –2). (4)

7. The length of a parametric curve is given by the formula

$$L = \int_b^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

The curve $x = 2\cos^3 t$, $y = 2\sin^3 t$ has four congruent sections as shown in the diagram.



Calculate the length of the curve. (6)

8. (*a*) Prove that one root of the polynomial $f(z) = z^4 - 8z^3 + 27z^2 - 50z + 50$ is $z = 3 + i$. (2)

(*b*) Hence find all the other roots. (4)

9. An open bin in the shape of a cylinder is to be constructed from 48 ft^2 of material.

Calculate the radius which gives maximum volume.

Hence find the maximum volume of the bin. (8)

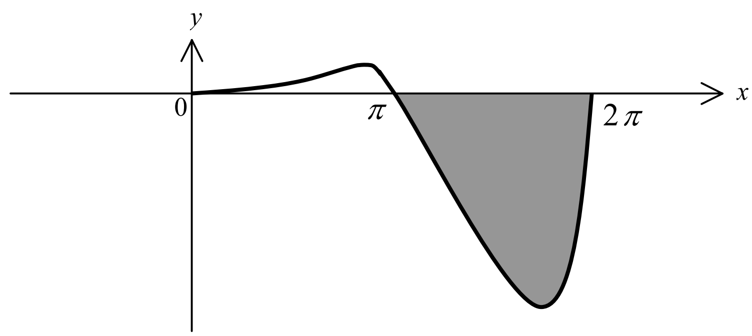
10. (*a*) Use integration by parts to find reduction formulae for $\int x^n \sin x \, dx$ and $\int x^n \cos x \, dx$. (6)

(*b*) Use these formulae to evaluate

$$\int x^3 \sin x \, dx,$$

simplifying your answer as far as possible. (4)

(c) Hence calculate the shaded area between the curve $y = x^3 \sin x$ and the x-axis, as shown in the diagram. (2)



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11. The function $f(x)$ is defined as

$$f(x) = \frac{8}{4-(x-1)^2}, \quad x \neq -1, x \neq 3, \ x \in \Re .$$

(*a*) Sketch the graph of $y = f(x)$, showing clearly its intersections with the axes and its turning points with appropriate justification. (8)

(*b*) Hence sketch the graph of the function $g(x)$, defined as

$$g(x) = \begin{cases} \dfrac{8}{4-(x-1)^2}, & x \leq 1 \\ x+1, & x > 1 \end{cases} \quad .$$ (2)

END OF QUESTION PAPER

Marking Scheme - AH Practice Paper C

	Give one mark for each •	Illustrations for awarding each mark
1(a)	**ans:** $6\tan 3x\sec^2 3x$ *or* $\dfrac{6\sin 3x}{\cos^3 3x}$ **3 marks** • knows how to differentiate • correct chain rule factor • answer	 • 2 tan3x • $\dfrac{d}{dx}\tan 3x = 3\sec^2 3x$ • $6\tan 3x\sec^2 3x$ *or* $\dfrac{6\sin 3x}{\cos^3 3x}$
1(b)	**ans:** $\dfrac{2x-12x^2}{(3x+1)^7} = -\dfrac{2x(6x-1)}{(3x+1)^7}$ **3 marks** • knows how to differentiate quotient • all derivatives correct • simplifies answer	 • $f'(x) = \dfrac{2x(3x+1)^6 - x^2\cdot 3\cdot 6(3x+1)^5}{(3x+1)^{12}}$ • as above • $\dfrac{2x-12x^2}{(3x+1)^7} = -\dfrac{2x(6x-1)}{(3x+1)^7}$
2.	**ans:** $\dfrac{23}{30}$ **4 marks** • splits up decimal • makes geometric series • finds sum to infinity of geometric series • answer	 • $0{\cdot}76666666\ldots = 0{\cdot}7 + 0{\cdot}066666666\ldots$ • $0{\cdot}7 + \dfrac{6}{100} + \dfrac{6}{1000} + \dfrac{6}{10000} + \ldots$ • $0{\cdot}7 + \dfrac{\frac{6}{100}}{1-\frac{1}{10}}$ • $\dfrac{69}{90} = \dfrac{23}{30}$
3.	**ans:** proof **5 marks** • show true for $n = 1$ • state inductive hypothesis • consider the case for $n = k + 1$ • carry out manipulation • state conclusion	 • $\left\{ \begin{array}{l} LHS = 1\times 1! = 1;\ RHS = (1+1)! - 1 = 2! - 1 = 1 \\ \quad\quad So\ true\ when\ n = 1 \end{array} \right.$ • Assume $1\times 1! + 2\times 2! + \ldots + k\times k! = (k+1)! - 1$ • Consider $\sum_{r=1}^{k+1} r\times r!$ • $\sum_{r=1}^{k+1} r\times r! = \ldots = [(k+1)+1]! - 1$ • So, if the formula is true for n, it is valid for $n + 1$. Since it is valid for $n = 1$, it is therefore true for all $n \geq 1$.

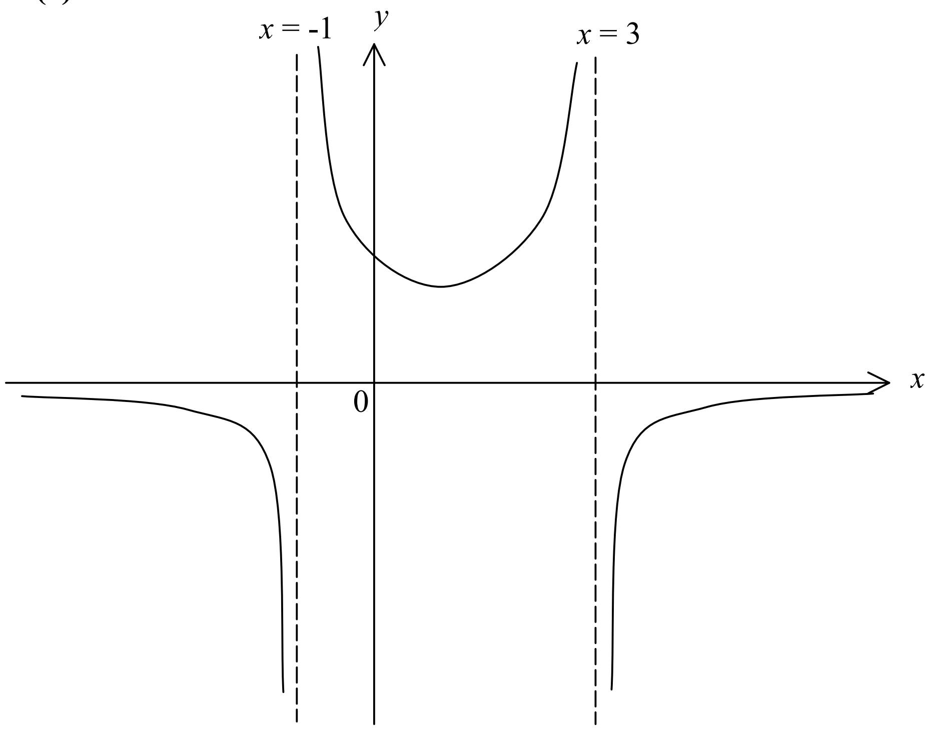
	Give one mark for each •	Illustrations for awarding each mark
4.	**ans:** $\frac{1}{2}\ln 3$ or $\ln\sqrt{3}$ **8 marks** • know how to find partial fractions • finds A and B • proves result • substitutes expression into integral • integrates terms correctly • deals with logs correctly • substitutes limits • answer	• $\frac{1}{1-4x^2} = \frac{A}{1-2x} + \frac{B}{1+2x}$ • $A = ½$, $B = ½$ • $\frac{1}{2}\left[\frac{1}{1-2x} + \frac{1}{1+2x}\right] = \frac{1}{2}\left[\frac{1}{2x+1} - \frac{1}{2x-1}\right]$ • $\frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}}\left[\frac{1}{2x+1} - \frac{1}{2x-1}\right]dx$ • $\frac{1}{2}\left[\frac{1}{2}\ln\lvert 2x+1\rvert - \frac{1}{2}\ln\lvert 2x-1\rvert\right]$ • $\frac{1}{4}\ln\frac{\lvert 2x+1\rvert}{\lvert 2x-1\rvert}$ • $\frac{1}{4}\ln 3 - \frac{1}{4}\ln\frac{1}{3}$ • $\frac{1}{4}\ln 9 = \frac{1}{2}\cdot\frac{1}{2}\ln 9 = \frac{1}{2}\ln 3$
5(a)	**ans:** $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ **3 marks** • correct coefficients and correctly deals with i • correct powers • correct expression	• $\cos^3\theta + 3\cos^2\theta\cdot i\sin\theta + 3\cos\theta\cdot(i\sin\theta)^2 + (i\sin\theta)^3$ • as above • $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
5(b)	**ans:** $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$ **4 marks** • uses De Moivre's Theorem correctly • equates imaginery parts • manipulates formulae • answer	• $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ • $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ • $\sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin 3\theta$ • $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

	Give one mark for each •	Illustrations for awarding each mark
6.	**ans:** $y = x - 4$ **4 marks** • knows to differentiate implicitly • finds dy/dx • finds gradient • finds equation of line	 • $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$ • $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$ • $m = 1$ • $y = x - 4$
7.	**ans:** 12 units **6 marks** • chooses correct limits of integration • finds $\frac{dx}{dt}, \frac{dy}{dt}$. • substitutes $\frac{dx}{dt}, \frac{dy}{dt}$ correctly into formula • correct manipulation to simplify integral • integrates correctly • answer	 • 0 and $\frac{\pi}{2}$ • $\frac{dx}{dt} = -6\cos^2 t \cdot \sin t$; $\frac{dy}{dt} = 6\sin^2 t \cdot \cos t$ • $\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $= \int_0^{\frac{\pi}{2}} \sqrt{(-6\cos^2 t \sin t)^2 + (6\sin^2 t \cos t)^2}\, dt$ • $\int_0^{\frac{\pi}{2}} 6\cos t \sin t\ dt$ • $\left[3\sin^2 t\right]_0^{\frac{\pi}{2}} = 3$ • $4 \times 3 = 12$ units
8(a)	**ans:** proof **2 marks** • finds higher powers of z • substitutes to prove $f(3+i) = 0$	 • $z^2 = 8 + 6i,\ z^3 = 18 + 26i,\ z^4 = 28 + 96i$ • proof
8(b)	**ans:** $1+2i, 1-2i, 3+i, 3-i$ **4 marks** • states 3 – i is root • finds quadratic factor • divides polynomial to find other quadratic factor • finds other roots	 • • $(z-(3+i))(z-(3-i)) = z^2 - 6z + 10$ • $z^4 - 8z^3 + 27z^2 - 50z + 50 \div (z^2 - 6z + 10)$ $= z^2 - 2z + 5$ • $1+2i, 1-2i, 3+i, 3-i$

	Give one mark for each •	Illustrations for awarding each mark
9.	**ans:** $36\cdot1$ ft^3 **8 marks** • for surface area and volume equations (stated or implied) • re-writes surface area formula in terms of h • substitutes h into volume formula • knows to differentiate volume and solve derivative = 0 • differentiates correctly w.r.t. r • finds r • checks nature • finds maximum volume	• $S.A. = \pi r^2 + 2\pi rh = 48$ $V = \pi r^2 h$ • $h = \dfrac{48 - \pi r^2}{2\pi r}$ • $V = 24r - \dfrac{1}{2}\pi r^3$ • strategy • $V' = 24 - \dfrac{3}{2}\pi r^2 = 0$ • $r = \sqrt{\dfrac{16}{\pi}} = \dfrac{4}{\sqrt{\pi}} \approx 2\cdot26$ • nature table (showing maximum) • $64/\sqrt{\pi} \approx 36\cdot1$
10(a)	**ans:** $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$ $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$ **6 marks** • chooses correct u and v each time • differentiates u correctly • integrates v • knows how to use integration by parts • 1st formula correct • 2nd formula correct	• $u = x^n$; $v = \sin x$ or $v = \cos x$ • $u' = nx^{n-1}$ • $\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ • strategy • $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$ $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$ $+ C$ •

	Give one mark for each •	Illustrations for awarding each mark
10(b)	**ans:** $x(6-x^2)\cos x+3(x^2-2)\sin x+C$ **4 marks** • first application of formulae • second application of formulae • third application of formulae • simplification	• $-x^3\cos x+3\int x^2\cos x\,dx$ • $-x^3\cos x+3\left[x^2\sin x-2\int x\sin x\,dx\right]$ • $-x^3\cos x+3x^2\sin x-6\left[-x\cos x+\int\cos x\,dx\right]$ • $x(6-x^2)\cos x+3(x^2-2)\sin x+C$
10(c)	**ans:** 222·5 units^2 **2 marks** • knows to substitute limits into answer • finds area	• $2\pi(6-(2\pi)^2)\cos 2\pi+3((2\pi)^2-2)\sin 2\pi$ $-\ (\pi(6-\pi^2)\cos\pi+3(\pi^2-2)\sin\pi)$ • $-222{\cdot}5$ so area = 222·5 units^2
11(a)	**ans:** sketch **8 marks** • finds equations of vertical asymptotes • finds equation of horizontal asymptote • finds y-intercept and attempt to solve $y=0$ • differentiates • sets $f'(x)=0$ • finds coordinate of turning point • justifies nature • sketch of graph	• $4-(x-1)^2=0\Rightarrow x=-1,\ x=3$ • $y=0$ • $\left(0,\frac{8}{3}\right)$ • $f'(x)=\frac{16(x-1)}{(x-3)^2(x+1)^2}$ • $f'(x)=\frac{16(x-1)}{(x-3)^2(x+1)^2}=0$ • $x=1,\ y=2$ i.e. (1, 2) • nature table or 2nd derivative – minimum • see sketch
11(b)	**ans:** sketch **2 marks** • knows to draw first graph up to $x=1$ • adds sketch of $y=x+1$ for $x>1$	• sketch on next page •

Sketch for question 11(a)



Sketch for question 11(b)