Practice Examination A (Unit 3)
(Assessing only Unit 3)

MATHEMATICS
Advanced Higher Grade

Time allowed - 1 hour

Read Carefully

1. Full credit will be given only where the solution contains appropriate working.
2. **Calculators may be used in this paper.**
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

All questions should be attempted

1. (*a*) Evaluate the product of the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(2)

(*b*) Find the general result of the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(2)

(*c*) Find the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ given that

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 - 3xy + 4y^2 \text{ and } \begin{pmatrix} p & q \\ r & s \end{pmatrix}^2 = \begin{pmatrix} -1 & 14 \\ -35 & 6 \end{pmatrix}.$$

(4)

2. Prove by induction that $2^{3n-1} + 3$ is divisible by 7 for all positive integers *n*. (5)

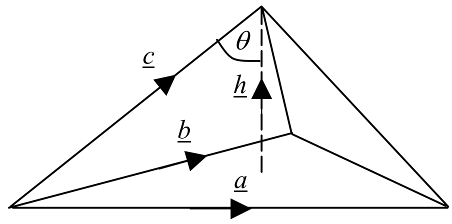
3. (*a*) Find the first five non-zero terms of the Maclaurin series for $\ln(1+x)$. (5)

(*b*) Deduce the Maclaurin series for $\ln(1-2x)$. (2)

(*c*) Hence find the first five terms of the Maclaurin series for $\ln(1-x-2x^2)$. (3)

4. (*a*) Prove that the volume of a tetrahedron is given by the formula

Volume = $\frac{1}{6}\left|\underline{a} \times \underline{b} \cdot \underline{c}\right|$ (4)



(*b*) Find the volume of tetrahedron OABC where O is the origin and A, B and C are the points (3, 2, 4), (4, 3, 5) and (0, 5, 3). (4)

5. (*a*) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}.$$

(6)

(*b*) Hence determine the solution which satisfies the conditions $y(0) = 1,\ y'(0) = 3$. (4)

End of Question Paper

Marking Scheme - AH – Practice Examination B (Unit 3)

	Give one mark for each •	Illustrations for awarding each mark
1(a)	**ans:** $x^2+3xy+2y^2$ **2 marks** • multiplies first 2 matrices together correctly • multiplies final 2 correctly	• $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} x-3y & 6x+2y \end{pmatrix}$ • $\begin{pmatrix} x-3y & 6x+2y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2+3xy+2y^2$
1(b)	**ans:** $ax^2+(b+c)xy+dy^2$ **2 marks** • multiplies first 2 matrices together correctly • multiplies final 2 correctly	• $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax+cy & bx+dy \end{pmatrix}$ • $\begin{pmatrix} ax+cy & bx+dy \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2+bxy+cxy+dy^2$
1(c)	**ans:** $\begin{pmatrix} 3 & 2 \\ -5 & 4 \end{pmatrix}$ **4 marks** • states p and s • squares matrix • equates coefficients • finds q and r	• $p=3,\ s=4$ • $\begin{pmatrix} 3 & q \\ r & 4 \end{pmatrix}\begin{pmatrix} 3 & q \\ r & 4 \end{pmatrix} = \begin{pmatrix} 9+qr & 7q \\ 7r & rq+16 \end{pmatrix}$ • $7q = 14$, $7r = -35$ • $q = 2$, $r = -5$
2.	**ans:** Proof **5 marks** • show true for $n = 1$ • state inductive hypothesis • consider the case for $n = k + 1$ • carry out manipulation • state conclusion	• $2^{3(1)-1}+3 = 2^2+3 = 7$ which is divisible by 7 so true for $n = 1$ • Assume $2^{3k-1}+3 = 7m$ for some m $\in$ **N** • Consider $2^{3(k+1)-1}+3 = 2^{3k+2}+3$ • $2^{3k+2}+3 = \ldots = 8\left(2^{3k-1}+3\right)-21$ $= 8(7m)-21 = 7(8m-3)$ which is divisible by 7 • So, if the formula is valid for n, it is valid for $n + 1$. Since it is valid for $n = 1$, it is therefore true for all $n \geq 1$.

	Give one mark for each •	Illustrations for awarding each mark
3(a)	**ans:** $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$ **5 marks** • finds values for $f(0),\ f'(0)$ • finds value for $f''(0)$ • finds values for higher order derivatives • method • final statement of terms	 • $f(0)=0,\ f'(0)=1$ • $f''(0)=-1$ • 2, -6, -24 • $f(x)\approx f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots$ • $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots$
3(b)	**ans:** $-2x-2x^2-\frac{8}{3}x^3-4x^4-\frac{32}{5}x^5$ **2 marks** • knows to replace *x* in (*a*) with (-2*x*) • all calculations correct	 • $-2x-\frac{(-2x)^2}{2}+\frac{(-2x)^3}{3}-\frac{(-2x)^4}{4}+\frac{(-2x)^5}{5}$ • $-2x-2x^2-\frac{8}{3}x^3-4x^4-\frac{32}{5}x^5$
3(c)	**ans:** $-x-\frac{5}{2}x^2-\frac{7}{3}x^3-\frac{17}{4}x^4-\frac{31}{5}x^5$ **3 marks** • recognises connection between series • knows how to combine series • combines series correctly	 • $1-x-2x^2=(1+x)(1-2x)$ • $\ln(1-x-2x^2)=\ln(1+x)+\ln(1-2x)$ • $-x-\frac{5}{2}x^2-\frac{7}{3}x^3-\frac{17}{4}x^4-\frac{31}{5}x^5$
4(a)	**ans:** Proof **4 marks** • knows how to calculate volume • calculates area of base correctly • calculates height correctly • proves formula	 • Volume = $\frac{1}{3}$ area of base × height • $\frac{1}{2}\lvert \underline{a}\times\underline{b}\rvert$ • $\lvert \underline{c}\rvert\cos\theta$ • $\frac{1}{6}\lvert \underline{a}\times\underline{b}\cdot\underline{c}\rvert$
4(b)	**ans:** $1\frac{1}{3}$ units^3 **4 marks** • knows how to calculate vector product • calculates vector product correctly • calculates scalar product correctly • answer	 • $\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & 2 & 4\\ 4 & 3 & 5\end{vmatrix}$ • $-2\underline{i}+\underline{j}+\underline{k}$ • $\begin{pmatrix}-2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\5\\3\end{pmatrix}=8$ • Volume = $\frac{1}{6}\times 8=1\frac{1}{3}$ units^3

	Give one mark for each •	Illustrations for awarding each mark
5(a)	**ans:** $y = Ae^{-2x} + Be^{-x} - 2xe^{-2x}$ **6 marks** • solves auxiliary equation • finds complimentary function • states correct form of Particular Integral • calculates 1st and 2nd derivatives of PI • calculates value of *C* • states general solution	• $m^2 + 3m + 2 = 0 \Rightarrow m = -2$, $m = -1$ • $y = Ae^{-2x} + Be^{-x}$ • $y = Cxe^{-2x}$ • $\frac{dy}{dx} = Ce^{-2x} - 2Cxe^{-2x}, \frac{d^2y}{dx^2} = -4Ce^{-2x} + 4Cxe^{-2x}$ • $C = -2$ • $y = Ae^{-2x} + Be^{-x} - 2xe^{-2x}$
5(b)	**ans:** $y = 7e^{-x} - 2e^{-2x}(3 + x)$ **4 marks** • calculates derivative of general solution • substitutes initial conditions into equations • solves equations simultaneously • states solution	• $y' = -2Ae^{-2x} - Be^{-x} - 2e^{-2x} + 4xe^{-2x}$ • $1 = A + B$, $5 = -2A - B$ • $A = -6$, $B = 7$ • $y = 7e^{-x} - 6e^{-2x} - 2xe^{-2x}$

Total 41 marks