

**Practice Examination B**  
*(Assessing Units 1 & 2)*

**MATHEMATICS**  
**Advanced Higher Grade**

**Time allowed - 2 hours 30 minutes**

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Read Carefully

1. Full credit will be given only where the solution contains appropriate working.
2. **Calculators may be used in this paper.**
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

**All questions should be attempted**

1. Evaluate  $\int_{-1}^0 \frac{dx}{\sqrt{3-2x-x^2}}$ . (5)

2. Verify that  $2i$  is a solution of  $z^4 - 4z^3 + 17z^2 - 16z + 52 = 0$ .  
Hence find all the solutions. (5)

3. Use Gaussian Elimination to solve the following system of equations.

$$\begin{aligned}x - y + 2z &= 7 \\3x + 2y + z &= -9 \\2x + y - 3z &= -20\end{aligned}$$

(5)

4. (a) Write the binomial expansion of  $(a + b)^4$ . (2)

(b) Find  $\left(x + \frac{2}{x}\right)^4 - \left(x - \frac{2}{x}\right)^4$  in its simplest form. (5)

5. (a) Differentiate  $f(x) = e^{\frac{x+1}{x-1}}$ ,  $x > 1$ . (4)

(b) Find the equation of the tangent to the curve  $2x^2 - 3xy + y^2 = 0$  at the point  $(1, 1)$ . (4)

6. Let  $z = 3 - i$  and let  $\bar{z}$  be the complex conjugate of  $z$ .

Solve  $a\left(\frac{z}{\bar{z}}\right) + bz = 22 - 14i$ , for  $a, b \in \mathbf{R}$ . (4)

7. If  $k$  is a positive integer and the coefficient of  $x^2$  in the expansion of  $(k - 4x)^6$  is 19440, find the value of  $k$ . (4)

8. The parametric equations  $x = \frac{t-3}{2(1+2t)}$ ,  $y = \frac{t}{2(1+2t)}$  represent a line, where  $t \in \mathbf{R}$ .

Find the Cartesian equation of the line, and show that the point  $(\frac{11}{2}, 1)$  lies on the line. (5)

9. The first three terms of an arithmetic series are  $8 + 16 + 24 + \dots$

(a) Find, in terms of  $n$ , an expression for  $u_n$ , the  $n^{\text{th}}$  term, and  $S_n$ , the sum to  $n$  terms. (4)

(b) Hence find the sum of the natural numbers that are both multiples of 8 and smaller than 1000. (3)

10. Express  $2 - 2\sqrt{3}i$  in polar form and hence find values for  $(2 - 2\sqrt{3}i)^{\frac{3}{2}}$ , writing your answers in the form  $p + qi$ , where  $p, q \in \mathbf{R}$ . (6)

11. By expressing  $0 \cdot 2\dot{3}$  as a geometric series, write  $0 \cdot 2\dot{3}$  in the form  $\frac{a}{b}$ , where  $a, b \in \mathbf{N}$ . (3)

12. Find the 4<sup>th</sup> roots of unity and show that the sum of these roots is zero. (5)

13. The integral  $I_n$  is given by

$$I_n = \int \sin^n x \, dx.$$

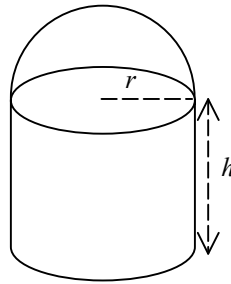
(a) By using the fact that  $\sin^n x = \sin x \sin^{n-1} x$ , prove the reduction formula

$$I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2} \quad (5)$$

(b) Use the above result to find a reduction formula for  $\int_0^{\pi/2} \sin^n x$ . (2)

(c) Hence evaluate  $\int_0^{\pi/2} \sin^8 x$ . (3)

14. A child's drinking cup is made in the shape of a circular cylinder with a hemispherical top.



The cylinder has height  $h$  cm and radius  $r$  cm, and the cup has a total surface area of  $80\pi$  cm<sup>2</sup>.

- (a) Find an expression for the height  $h$  in terms of the radius  $r$ .  
[ surface area of a sphere =  $4\pi r^2$  ] (2)
- (b) Find the values of  $h$  and  $r$  for which the cup has a maximum volume.  
Hence find the maximum volume. (8)
15. The function  $f(x)$  is given by  $f(x) = \frac{x^2 - 4}{x^2 + 8x}$ .
- (a) Write down the equations of the asymptotes of  $f(x)$ . (2)
- (b) Prove that  $f(x)$  has no stationary points. (3)
- (c) Sketch the curve of  $f(x)$ , showing clearly all its features. (3)
- (d) Using the sketch in part (c), construct the graph of  $\frac{1}{f(x)}$ , the curve of the reciprocal function. (4)

***End of Question Paper***

### Marking Scheme - AH Practice Paper B

	Give one mark for each •	Illustrations for awarding each mark
1.	<b>ans:</b> $\pi/6$ <span style="float: right;"><b>5 marks</b></span> <ul style="list-style-type: none"> <li>• knowing to complete the square</li> <li>• knowing to use a suitable substitution</li> <li>• finds new limits</li> <li>• integrates correctly</li> <li>• answer</li> </ul>	<ul style="list-style-type: none"> <li>• <math>\int_{-1}^0 \frac{dx}{\sqrt{4-(x+1)^2}}</math></li> <li>• <math>u = x + 1, du = dx,</math> <math>x = -1 : u = 0</math></li> <li>• <math>x = 0 : u = 1</math></li> <li>• <math>\sin^{-1} \frac{u}{2} \Big _0^1</math></li> <li>• <math>\pi/6</math></li> </ul>
2.	<b>ans:</b> $2i, -2i, 2 + 3i, 2 - 3i$ <span style="float: right;"><b>5 marks</b></span> <ul style="list-style-type: none"> <li>• substitutes <math>2i</math> correctly into equation and proves result</li> <li>• states complex conjugate is a solution</li> <li>• finds first quadratic factor</li> <li>• finds second quadratic factor by dividing original equation by first quadratic factor</li> <li>• uses quadratic formula to find other 2 solutions</li> </ul>	<ul style="list-style-type: none"> <li>• <math>(2i)^4 - 4(2i)^3 + 17(2i)^2 - 16(2i) + 52</math> <math>= 16 + 32i - 68 - 32i + 52 = 0</math></li> <li>• <math>-2i</math> is a solution</li> <li>• <math>(z - 2i)(z + 2i) = z^2 + 4</math></li> <li>• <math>(z^4 - 4z^3 + 17z^2 - 16z + 52) \div (z^2 + 4)</math> <math>= z^2 - 4z + 13</math></li> <li>• <math>2 + 3i, 2 - 3i</math></li> </ul>
3.	<b>ans:</b> $x = -3, y = -2, z = 4$ <span style="float: right;"><b>5 marks</b></span> <ul style="list-style-type: none"> <li>• method</li> <li>• first modified system</li> <li>• second modified system</li> <li>• method of back-substitution</li> <li>• values of <math>x, y</math> and <math>z</math></li> </ul>	<ul style="list-style-type: none"> <li>• <math>\begin{bmatrix} 1 &amp; -1 &amp; 2 &amp; 7 \\ 3 &amp; 2 &amp; 1 &amp; -9 \\ 2 &amp; 1 &amp; -3 &amp; -20 \end{bmatrix}</math></li> <li>• <math>\begin{bmatrix} 1 &amp; -1 &amp; 2 &amp; 7 \\ 0 &amp; 5 &amp; -5 &amp; -30 \\ 0 &amp; 3 &amp; -7 &amp; -34 \end{bmatrix}</math></li> <li>• <math>\begin{bmatrix} 1 &amp; -1 &amp; 2 &amp; 7 \\ 0 &amp; 5 &amp; -5 &amp; -30 \\ 0 &amp; 0 &amp; -4 &amp; -16 \end{bmatrix}</math></li> <li>• <math>-4z = -16 \Rightarrow z = 4</math></li> <li>• <math>y = -2, x = -3</math></li> </ul>

	Give one mark for each •	Illustrations for awarding each mark
4(a)	<b>ans:</b> $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ <b>2 marks</b> <ul style="list-style-type: none"> <li>• all powers in expansion</li> <li>• all coefficients in expansion</li> </ul>	<ul style="list-style-type: none"> <li>• and • <math>a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4</math></li> </ul>
4(b)	<b>ans:</b> $16x^2 + \frac{64}{x^2}$ <b>5 marks</b> <ul style="list-style-type: none"> <li>• correctly substituting <math>x</math> and <math>\frac{2}{x}</math> in place of <math>a</math> &amp; <math>b</math></li> <li>• correct expression</li> <li>• substitutes <math>x</math> and <math>-\frac{2}{x}</math> in place of <math>a</math> and <math>b</math></li> <li>• correct expression</li> <li>• answer</li> </ul>	<ul style="list-style-type: none"> <li>• <math>x^4 + 4x^3\left(\frac{2}{x}\right) + 6x^2\left(\frac{2}{x}\right)^2 + 4x\left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4</math></li> <li>• <math>x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}</math></li> <li>•</li> <li>• <math>x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4</math></li> <li>• <math>x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}</math></li> <li>• <math>16x^2 + \frac{64}{x^2}</math></li> </ul>
5(a)	<b>ans:</b> $-\frac{2}{(x-1)^2} e^{\frac{x+1}{x-1}}$ <b>4 marks</b> <ul style="list-style-type: none"> <li>• knowing how to differentiate <math>e</math></li> <li>• chain rule factor</li> <li>• using quotient rule correctly</li> <li>• answer</li> </ul>	<ul style="list-style-type: none"> <li>• <math>e^{\frac{x+1}{x-1}}</math></li> <li>• <math>\frac{d}{dx}\left(\frac{x+1}{x-1}\right)</math></li> <li>• <math>\frac{d}{dx}\left(\frac{x+1}{x-1}\right) = -\frac{2}{(x-1)^2}</math></li> <li>• <math>-\frac{2}{(x-1)^2} e^{\frac{x+1}{x-1}}</math></li> </ul>
5(b)	<b>ans:</b> $y = x$ <b>4 marks</b> <ul style="list-style-type: none"> <li>• differentiating correctly</li> <li>• finding expression for <math>\frac{dy}{dx}</math></li> <li>• finding gradient</li> <li>• finding equation of tangent</li> </ul>	<ul style="list-style-type: none"> <li>• <math>4x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0</math></li> <li>• <math>\frac{dy}{dx} = \frac{3y - 4x}{2y - 3x}</math></li> <li>• <math>m = 1</math></li> <li>• <math>y - 1 = 1(x - 1)</math></li> </ul>

	Give one mark for each •	Illustrations for awarding each mark
6.	<b>ans:</b> $a = 20$ , $b = 2$ <b>4 marks</b> <ul style="list-style-type: none"> <li>finding complex conjugate</li> <li>finding <math>\frac{z}{\bar{z}}</math></li> <li>equating real and imaginary parts</li> <li>Solving simultaneous equations</li> </ul>	<ul style="list-style-type: none"> <li><math>\bar{z} = 3 + i</math></li> <li><math>\frac{z}{\bar{z}} = \frac{4}{5} - \frac{3}{5}i</math></li> <li><math>\frac{4}{5}a + 3b = 22</math>, <math>-\frac{3}{5}a - b = -14</math></li> <li><math>a = 20</math>, <math>b = 2</math></li> </ul>
7.	<b>ans:</b> $k = 3$ <b>4 marks</b> <ul style="list-style-type: none"> <li>Finding general term</li> <li>knowing to put <math>r = 2</math></li> <li>substituting <math>r = 2</math> correctly</li> <li>finding value of <math>k</math></li> </ul>	<ul style="list-style-type: none"> <li><math>\binom{6}{r} k^{6-r} (-4x)^r</math></li> <li><math>r = 2</math></li> <li><math>\binom{6}{2} k^4 (-4)^2 = 19440 \Rightarrow 240k^4 = 19440</math></li> <li><math>k = 3</math></li> </ul>
8.	<b>ans:</b> $2x - 14y + 3 = 0$ <b>5 marks</b> <ul style="list-style-type: none"> <li>changing the subject of <math>x</math> to <math>t</math></li> <li>changing the subject of <math>y</math> to <math>t</math></li> <li>knowing to equate expressions</li> <li>finding equation of line</li> <li>proving point lies on line</li> </ul>	<ul style="list-style-type: none"> <li><math>t = \frac{3 + 2x}{1 - 4x}</math></li> <li><math>t = \frac{2y}{1 - 4y}</math></li> <li><math>\frac{3 + 2x}{1 - 4x} = \frac{2y}{1 - 4y}</math></li> <li><math>2x - 14y + 3 = 0</math></li> <li><math>2\left(\frac{11}{2}\right) - 14(1) + 3 = 11 - 14 + 3 = 0</math> as required</li> </ul>
9(a)	<b>ans:</b> $8n$ , $4n(n + 1)$ <b>4 marks</b> <ul style="list-style-type: none"> <li>knowing how to find general term</li> <li>finding general term</li> <li>knowing how to find sum to n terms</li> <li>finding sum to n terms</li> </ul>	<ul style="list-style-type: none"> <li><math>u_n = a + (n - 1)d</math></li> <li><math>u_n = 8 + (n - 1)8 = 8n</math></li> <li><math>S_n = \frac{n}{2}[2a + (n - 1)d]</math></li> <li><math>S_n = \frac{n}{2}[16 + (n - 1)8] = 4n(n + 1)</math></li> </ul>
9(b)	<b>ans:</b> 62000 <b>3 marks</b> <ul style="list-style-type: none"> <li>knows how to find <math>n</math></li> <li>finds <math>n</math> correctly</li> <li>finds sum of terms</li> </ul>	<ul style="list-style-type: none"> <li><math>8n &lt; 1000 \Rightarrow n &lt; 125</math></li> <li><math>n = 124</math></li> <li><math>S_{124} = 4 \times 124 (124 + 1) = 62000</math></li> </ul>

	Give one mark for each •	Illustrations for awarding each mark
10.	<b>ans:</b> $-8i, 8i$ <b>6 marks</b> <ul style="list-style-type: none"> <li>finds modulus</li> <li>finds argument</li> <li>writes complex number in polar form</li> <li>deals with power correctly in polar form</li> <li>finds first solution</li> <li>finds second solution</li> </ul>	<ul style="list-style-type: none"> <li><math>\sqrt{2^2 + (-2\sqrt{3})^2} = 4</math></li> <li><math>\tan \theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3} \Rightarrow \theta = -\frac{\pi}{3}</math></li> <li><math>4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right)</math></li> <li><math>4^{3/2}, \frac{3}{2}\left(-\frac{\pi}{3} + 2k\pi\right)</math></li> <li><math>k = 0 \Rightarrow -8i</math></li> <li><math>k = 0 \Rightarrow 8i</math></li> </ul>
11.	<b>ans:</b> $\frac{7}{30}$ <b>3 marks</b> <ul style="list-style-type: none"> <li>writes <math>0.2333333\dots</math> As a geometric series</li> <li>finds sum to infinity</li> <li>answer</li> </ul>	<ul style="list-style-type: none"> <li><math>\frac{2}{10} + \left(\frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots\right)</math></li> <li><math>S_{\infty} = \frac{a}{1-r} = \frac{\frac{3}{100}}{1-\frac{1}{10}} = \frac{1}{30}</math></li> <li><math>\frac{2}{10} + \frac{1}{30} = \frac{7}{30}</math></li> </ul>
12.	<b>ans:</b> $1, i, -1, -i$ <b>5 marks</b> <ul style="list-style-type: none"> <li>writes 1 in polar form</li> <li>takes 4<sup>th</sup> root correctly</li> <li>knows values of <math>k</math> to use</li> <li>finds roots</li> <li>proves roots sum to zero</li> </ul>	<ul style="list-style-type: none"> <li><math>1 = \cos 2k\pi + i\sin 2k\pi</math></li> <li><math>z = \cos \frac{2k\pi}{4} + i\sin \frac{2k\pi}{4},</math></li> <li><math>k = 0, 1, 2, 3</math></li> <li><math>z = 1, i, -1, -i</math></li> <li><math>1 + i + (-1) + (-i) = 0</math> as required</li> </ul>
13(a)	<b>ans:</b> Proof <b>5 marks</b> <ul style="list-style-type: none"> <li>knowing to use integration by parts</li> <li>correct application of integration by parts</li> <li>knows to use trigonometric identity</li> <li>combining terms</li> <li>answer</li> </ul>	<ul style="list-style-type: none"> <li><math>\int \sin x \sin^{n-1} x dx</math></li> <li><math>-\cos x \sin^{n-1} x - \int (n-1)\cos x \sin^{n-2} x (-\cos x) dx</math></li> <li><math>= -\cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x dx</math></li> <li><math>\cos^2 x = 1 - \sin^2 x</math></li> <li><math>[1 + (n-1)] \int \sin^n x dx = -\cos x \sin^{n-1} x</math> <math>+ (n-1) \int \sin^{n-2} x dx</math></li> <li><math>\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}</math></li> </ul>

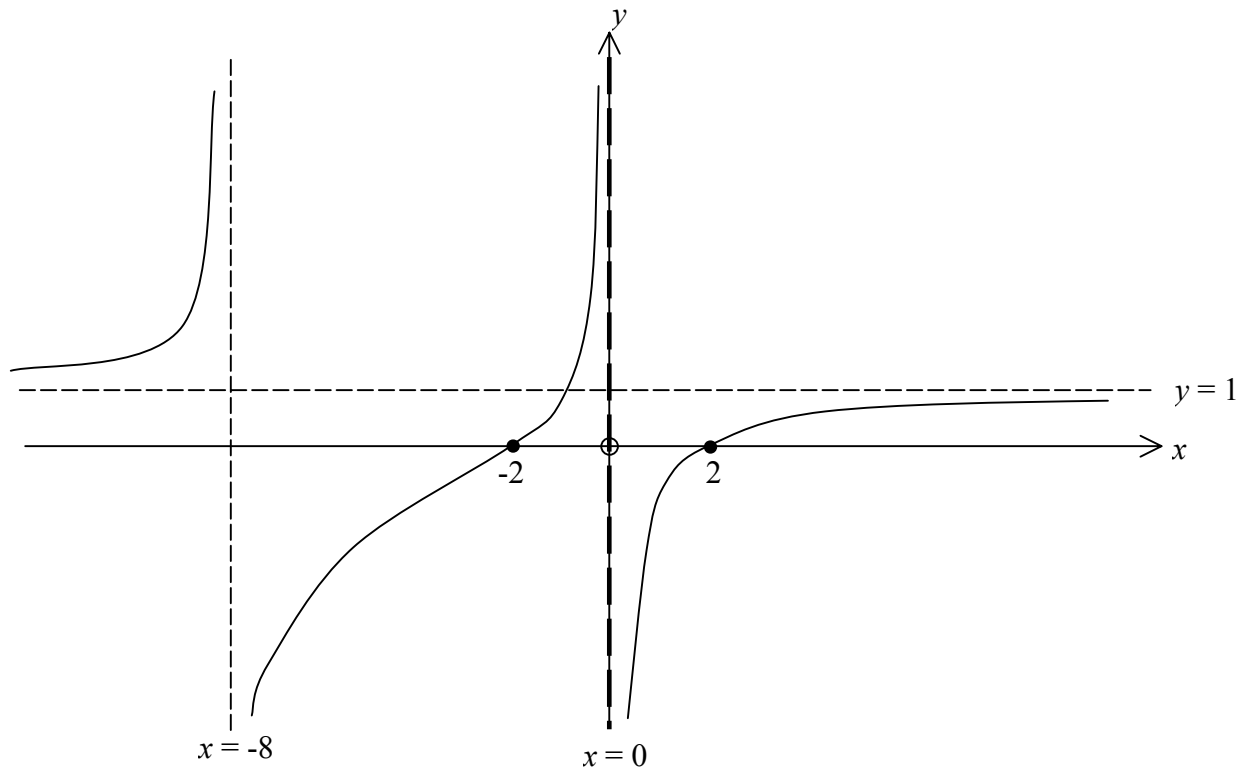


	Give one mark for each •	Illustrations for awarding each mark
13(b)	<b>ans:</b> $\frac{n-1}{n}I_{n-2}$ <b>2 marks</b> <ul style="list-style-type: none"> <li>applying limits to reduction formula</li> <li>answer</li> </ul>	<ul style="list-style-type: none"> <li><math>\left(-\frac{1}{n}\cos\frac{\pi}{2}\sin^{n-1}\frac{\pi}{2}\right) - \left(-\frac{1}{n}\cos 0\sin^{n-1}0\right)</math>  <math>+ \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx</math></li> <li><math>\frac{n-1}{n}I_{n-2}</math></li> </ul>
13(c)	<b>ans:</b> $\frac{35\pi}{256}$ <b>3 marks</b> <ul style="list-style-type: none"> <li>knowing to use reduction formula repeatedly</li> <li>correctly integrating <math>\sin^2 x</math></li> <li>answer</li> </ul>	<ul style="list-style-type: none"> <li><math>\int_0^{\pi/2} \sin^8 x dx = \frac{8-1}{8} \int_0^{\pi/2} \sin^6 x dx = \frac{7}{8}I_6</math> etc.</li> <li><math>\sin^2 x = \frac{1}{2}(1 - \cos 2x)</math></li> <li><math>\frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}</math></li> </ul>
14(a)	<b>ans:</b> $h = \frac{40}{r} - \frac{3r}{2}$ <b>2 marks</b> <ul style="list-style-type: none"> <li>finds surface area correctly</li> <li>finds <math>h</math> in terms of <math>r</math></li> </ul>	<ul style="list-style-type: none"> <li>cylinder = <math>2\pi rh + \pi r^2</math>, hemisphere = <math>2\pi r^2</math>  <math>\Rightarrow 80\pi = 2\pi rh + 3\pi r^2</math></li> <li><math>h = \frac{40}{r} - \frac{3r}{2}</math></li> </ul>
14(b)	<b>ans:</b> 4 cm, 4 cm, $80\pi \text{ cm}^3$ <b>8 marks</b> <ul style="list-style-type: none"> <li>finds correct expression for volume</li> <li>substitutes expression for <math>h</math></li> <li>differentiates volume</li> <li>puts derivative = 0</li> <li>finds value for <math>r</math></li> <li>checks nature is maximum</li> <li>finds value for <math>h</math></li> <li>finds value for <math>V</math></li> </ul>	<ul style="list-style-type: none"> <li><math>V = \pi r^2 h + \frac{2}{3}\pi r^3</math></li> <li><math>V = 40\pi r - \frac{5}{6}\pi r^3</math></li> <li><math>40\pi - \frac{5}{2}\pi r^2</math></li> <li><math>40\pi - \frac{5}{2}\pi r^2 = 0</math></li> <li><math>r = 4 \text{ cm}</math></li> <li>nature table</li> <li><math>h = 4 \text{ cm}</math></li> <li><math>V = \frac{320\pi}{3} \text{ cm}^3</math></li> </ul>
15(a)	<b>ans:</b> $x = 0, x = -8, y = 1$ <b>2 marks</b> <ul style="list-style-type: none"> <li>finds equations of vertical asymptotes</li> <li>finds equation of horizontal asymptote</li> </ul>	<ul style="list-style-type: none"> <li><math>x^2 + 8x = 0 \Rightarrow x = 0, x = -8</math></li> <li><math>f(x) = 1 - \frac{8x+4}{x^2+8x} \Rightarrow y = 1</math></li> </ul>

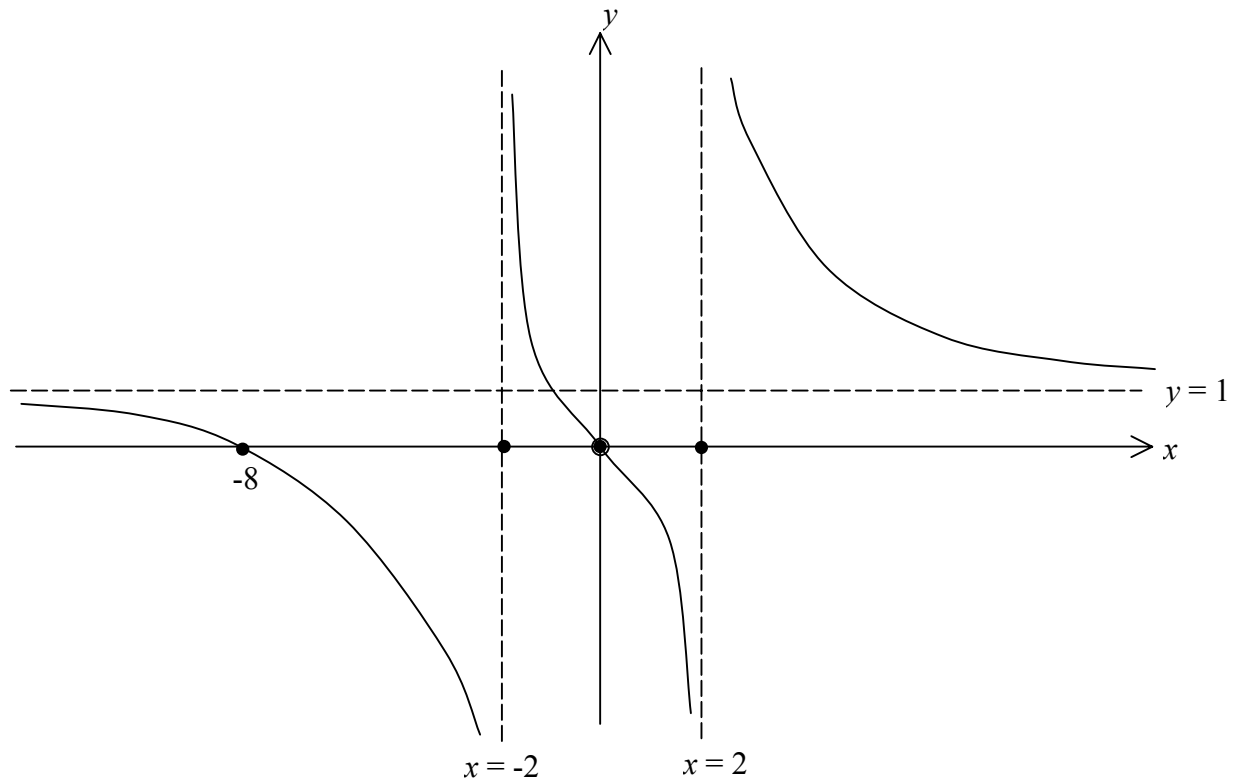
	<b>Give one mark for each •</b>	<b>Illustrations for awarding each mark</b>
15(b)	<b>ans:</b> Proof <span style="float: right;"><b>3 marks</b></span> <ul style="list-style-type: none"> <li>• calculates <math>f'(x)</math></li> <li>• knows to put <math>f'(x) = 0</math></li> <li>• uses discriminant to prove there is no solution</li> </ul>	<ul style="list-style-type: none"> <li>• and • <math>f'(x) = \frac{8x^2 + 8x + 32}{(x^2 + 8x)^2} = 0</math></li> <li>• <math>b^2 - 4ac = 64 - 4 \times 8 \times 32 &lt; 0 \therefore</math> no solution</li> </ul>
15(c)	<b>ans:</b> see graph on next page <span style="float: right;"><b>3 marks</b></span> <ul style="list-style-type: none"> <li>• <math>x</math> intercepts clearly shown</li> <li>• approaches asymptotes correctly</li> <li>• completing graph</li> </ul>	See next page
15(d)	<b>ans:</b> see graph on next page <span style="float: right;"><b>4 marks</b></span> <ul style="list-style-type: none"> <li>• asymptotes in (c) become roots</li> <li>• roots in (c) become asymptotes</li> <li>• approaches to asymptotes correct</li> <li>• completing graph</li> </ul>	See next page

**Total 96 marks**

15(c)



15(d)



**Total Marks: 96**