Practice Examination A
(Assessing Units 1 & 2)

MATHEMATICS
Advanced Higher Grade

Time allowed - 2 hours 30 minutes

Read Carefully

1. Full credit will be given only where the solution contains appropriate working.
2. **Calculators may be used in this paper.**
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

All questions should be attempted

1. Differentiate with respect to x, simplifying your answer as far as possible:

(*a*) $y = tan^{-1}\left(\dfrac{x+1}{x-1}\right)$ (4)

(*b*) $y = ln(sec\, x)$ (3)

2. Use Gaussian Elimination to solve the system

$$\begin{aligned} 2x + 3y - 4z &= -3 \\ x + 2y + 3z &= 3 \\ 3x - y - z &= 6 \end{aligned}$$ (5)

3. Prove by induction $\dfrac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ for all positive integers, n. (5)

4. Using the substitution $x = \sqrt{t}$, evaluate the integral

$$\int_{1/3}^{3} \frac{1}{t + \sqrt{t}}\, dt$$ (6)

5. Find the coefficient of x^5 in the expansion of $\left(x^3 + \dfrac{2}{x}\right)^7$. (3)

6. (*a*) Find partial fractions for $\dfrac{2x^2 + 6x + 36}{(x^2 + 9)(x + 3)}$. (4)

(*b*) Hence evaluate the integral $\displaystyle\int_{-2}^{0} \frac{2x^2 + 6x + 36}{(x^2 + 9)(x + 3)}\, dx$. (3)

7. Suppose that x and y are differentiable functions of t and that

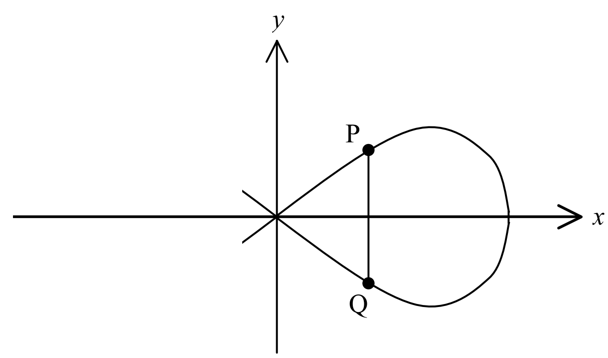
$$\frac{d^2y}{dx^2} = t^2 + 1 \quad , \quad \frac{dy}{dx} = t^3 + 3t \, .$$

Find $x(t)$ given that $x(1) = 4$. (6)

8. PQ is a chord of the loop of the curve $y^2 = x^2\,(8 - x^2), x > 0$.

PQ is parallel to the y-axis.

Calculate the maximum possible length of PQ.



(6)

9. (a) Find two numbers x and y whose sum is 4 and whose product is 8. (4)

(b) Plot the solutions on an Argand diagram. (2)

10. Use integration by parts to show that

$$\int x^3 \cos x \, dx \;=\; 3(x^2 - 2)\cos x \;+\; (x^3 - 6x)\sin x + C \quad .$$ (5)

11. (a) Find an expression for the sum of n terms of the series

$$2 + \tfrac{2}{3} + \tfrac{2}{9} + \ldots$$

in its simplest form. (4)

(b) If $S_n = \dfrac{242}{81}$, find the value of n. (2)

12. An investor has £2000 with which to open an account and plans to add a further £1000 each year.

All funds in the account will earn compound interest at a rate of 10% p.a. .

Let $x(t)$ be the amount of money in the account at time t years.

(*a*) Write down a first order differential equation representing the rate of change of money in the account each year. (2)

(*b*) Hence show that $t = 10\, ln\, \dfrac{(1000 + 0 \cdot 1x)}{1200}$. (7)

(*c*) How many years would it take to save £100 000 ? (2)

13. A function $f(x)$ is defined by

$$f(x) = \left| \frac{x^2 - 2x + 2}{x - 1} \right| .$$

(*a*) Write down the equation of the vertical asymptote of $f(x)$. (1)

(*b*) For the function $g(x) = \dfrac{x^2 - 2x + 2}{x - 1}$, show that there is a non-vertical asymptote and find its equation. (3)

(*c*) Find the coordinates of the stationary points of $g(x)$ and determine their nature. (5)

(*d*) By first considering the graph of $g(x)$, sketch the graph of $f(x)$ showing all its main features. (4)

14. The semi-circle $y = \sqrt{a^2 - x^2}$ is rotated about the x-axis to generate a sphere.

(*a*) Find an expression for the volume of the sphere. (8)

(*b*) Find the volume of the sphere with equation $y = \sqrt{25 - x^2}$. (2)

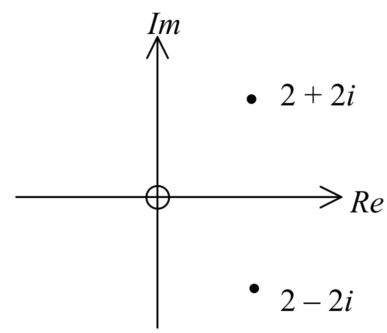
END OF QUESTION PAPER

Marking Scheme - AH Practice Paper A

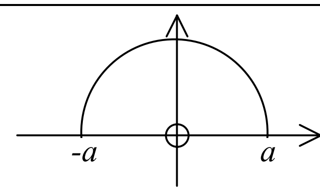
	Give one mark for each •	Illustrations for awarding each mark
1(a)	**ans:** $\frac{dy}{dx} = -\frac{1}{x^2+1}$ **4 marks**	
	• know how to differentiate $\tan^{-1}$	• $\frac{1}{1+\left(\frac{x+1}{x-1}\right)^2}$
	• chain rule factor	• $-\frac{2}{(x-1)^2}$
	• manipulating algebra	• $\frac{(x-1)^2}{2x^2+2} \times -\frac{2}{(x-1)^2}$
	• answer in simplest form	• $-\frac{1}{x^2+1}$
1(b)	**ans:** $\frac{dy}{dx} = \tan x$ **3 marks**	
	• know how to differentiate log	• $\frac{1}{\sec x}$
	• chain rule factor	• $\sec x \ \tan x$
	• answer in simplest form	• $\tan x$
2.	**ans:** (2, -1, 1) **5 marks**	
	• write system as an augmented matrix with 1 in top left-hand corner (optional)	• $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 2 & 3 & -4 & -3 \\ 3 & -1 & -1 & 6 \end{array}\right]$
	• first modified system	• $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & -7 & -10 & -3 \end{array}\right]$
	• second modified system	• $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & 0 & 60 & 60 \end{array}\right]$
	• using back-substitution to find z	• $z = 1$
	• using back-substitution to find x and y	• $y = -1$, $x = 2$

	Give one mark for each •	Illustrations for awarding each mark
3.	**ans:** proof by induction **5 marks** • show true for $n = 1$ • state inductive hypothesis • consider the case for $n = k + 1$ • carry out manipulation • state conclusion	• $\begin{cases} LHS = \frac{d}{dx}(x) = 1;\ RHS = 1 \times x^{1-1} = 1 \\ So\ true\ when\ n = 1 \end{cases}$ • Assume $\frac{d}{dx}\left(x^k\right) = k\,x^{k-1}$ • Consider $\frac{d}{dx}\left(x^{k+1}\right)$ • $\frac{d}{dx}\left(x \cdot x^k\right) = x^k + x \cdot kx^{k-1} = x^k + kx^k$ $= (k+1)x^k$ • So, if the formula is valid for n, it is valid for n+1. Since it is valid for $n = 1$, it is therefore true for all $n \geq 1$.
4.	**ans:** $ln\ 3$ **7 marks** • rewrite integral in terms of x • correct limits • tidy up integral • integrate • evaluate limits • manipulate surds • final answer	• and • $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2x}{x^2 + x}\,dx$ • $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2}{x+1}\,dx$ • $2\,ln(x+1)\Big]_{1/\sqrt{3}}^{\sqrt{3}}$ • $2\,ln\left(\sqrt{3}+1\right) - 2\,ln\left(\frac{1}{\sqrt{3}}+1\right)$ • and • $2\,ln\left(\frac{\sqrt{3}+1}{\frac{1}{\sqrt{3}}+1} \times \frac{\frac{1}{\sqrt{3}}-1}{\frac{1}{\sqrt{3}}-1}\right)$ $= 2\,ln\left(-\frac{3}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)\right) = 2\,ln\,\sqrt{3} = ln\ 3$
5.	**ans:** 560 **3 marks** • correct general term • put power of x equal to 5 and solve for r • calculate coefficient	• $\binom{7}{r}\left(x^3\right)^{7-r}\left(\frac{2}{x}\right)^r = \binom{7}{r}2^r\,x^{21-4r}$ • $21 - 4r = 5;\ r = 4$ • $\binom{7}{4}2^4 = 35 \times 16 = 560$

	Give one mark for each •	Illustrations for awarding each mark
6(a)	**ans:** $\dfrac{6}{x^2+9}+\dfrac{2}{x+3}$ **4 marks** • know how to find partial fractions • know how to find A, B and C • finds A • finds B and C	 • $\dfrac{Ax+B}{x^2+9}+\dfrac{C}{x+3}$ • $2x^2+6x+36=(x+3)(Ax+B)+C(x^2+9)$ • $A=0$ • $B=6$ and $C=2$
6(b)	**ans:** 3·37 units^2 **5 marks** • knows to express integral in partial fractions • and • integrates terms correctly • evaluates limits • final answer	 • $\int_{-2}^{0}\left(\dfrac{6}{x^2+9}+\dfrac{2}{x+3}\right)dx$ • and • $2\tan^{-1}\frac{x}{3}+2\ln\lvert x+3\rvert$ • $2\tan^{-1}0+2\ln 3-\left(2\tan^{-1}\left(-\frac{2}{3}\right)+2\ln 1\right)$ • 3·37 units^2
7.	**ans:** $x(t)=3t+1$ **6 marks** • knows formula for $\dfrac{d^2y}{dx^2}$ in parametric form • finds $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ • substitutes information into formula • finds $\frac{dx}{dt}$ in simplest form • integrates $\frac{dx}{dt}$ to find x • finds constant of integration	 • $\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ • $3t^2+3$ • $t^2+1=\dfrac{3t^2+3}{\frac{dx}{dt}}$ • 3 • $x(t)=\int 3\,dt=3t+c$ • $x(1)=4$; $c=1$
8.	**ans:** 8 units **6 marks** • knows to find max. and min. turning points • knows to use implicit differentiation • differentiates correctly • finds x-coordinate of relevant turning point • finds corresponding y-coordinates • finds max. distance	 • • • $\dfrac{dy}{dx}=\dfrac{2x(4-x^2)}{y}$ • x = -2, 0 or 2 and chooses x =2 from diagram • y = -4 or 4 • 8

	Give one mark for each •	Illustrations for awarding each mark
9(a)	**ans:** $2 + 2i$, $2 - 2i$ **4 marks** • set up system of equations • use substitution to obtain quadratic • use quadratic formula to solve quadratic • correct answer	 • $x + y = 4$; $xy = 8$ • $x^2 - 4x + 8 = 0$ • $x = \dfrac{4 \pm \sqrt{16 - 4(1)(8)}}{2}$ • $x = 2 + 2i$ or $x = 2 - 2i$
9(b)	**ans:** Diagram **2 marks** • Argand diagram correctly labelled • both points plotted and labelled	
10.	**ans:** Proof **5 marks** • first application of integration by parts • second application of integration by parts • knowing to use integ. by parts again • third application of integration by parts • answer in required form	 • $x^3 \sin x - \int 3x^2 \sin x \, dx$ • and • $x^3 \sin x - \left[-3x^2 \cos x + \int 6x \cos x \, dx\right]$ $= x^3 \sin x + 3x^2 \cos x - \int 6x \cos x \, dx$ • $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$ • $3(x^2 - 2)\cos x + (x^3 - 6x)\sin x + C$
11(a)	**ans:** $3\left(1 - \dfrac{1}{3^n}\right)$ **4 marks** • correct ratio • using correct formula • substituting correctly into formula • answer in simplest form	 • $r = \frac{1}{3}$ • $S_n = \dfrac{a(1 - r^n)}{1 - r}$ • $\dfrac{2\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} = \dfrac{2\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}}$ • $3\left(1 - \dfrac{1}{3^n}\right)$

	Give one mark for each •	Illustrations for awarding each mark
11(b)	**ans:** $n = 5$ **3 marks** • use formula correctly • manipulate formula • answer	 • $\frac{242}{81} = 3\left(1 - \frac{1}{3^n}\right) \Rightarrow \frac{242}{243} = 1 - \frac{1}{3^n}$ • $3^n = 243$ • $n = 5$ (using logs or trial and error)
12(a)	**ans:** $\frac{dx}{dt} = 1000 + 0 \cdot 1x$ **2 marks** • amount of money going into account each year • interest @ 10%	 • 1000 • $0{\cdot}1x$
12(b)	**ans:** $t = 10ln\frac{1000 + 0 \cdot 1x}{1200}$ **7 marks** • know to use method of separating variables • separates variables correctly • integrates LHS correctly • integrates RHS correctly (incl. constant of integration) • correct initial conditions • finds correct value of C • finds required solution	 • and • $\int \frac{dx}{1000+0{\cdot}1x} = \int dt$ • and • $10ln(1000 + 0 \cdot 1x) = t + C$ • $x = 2000$ at $t = 0$ • $C = 10ln1200$ • $t = 10ln\frac{1000 + 0 \cdot 1x}{1200}$
12(c)	**ans:** 23 years **2 marks** • substitute in value for x • answer	 • $t = 10ln\frac{1000+0{\cdot}1\times100000}{1200} = 10ln\frac{11000}{1200}$ • $22{\cdot}16$ years $\approx$ 23 years
13(a)	**ans:** $x = 1$ **1 mark** • states equation of vertical asymptote	 • $x = 1$
13(b)	**ans:** $y = x - 1$ **3 marks** • knows to divide • restating function • correctly stating equation of asymptote	 • and • $\frac{x^2 - 2x + 2}{x - 1} = (x - 1) + \frac{1}{x - 1}$ • $y = x - 1$

	Give one mark for each •	**Illustrations for awarding each mark**
13(c)	**ans:** Max at (0, -2), Min at (2, 2) **5 marks** • knows to find $\frac{dy}{dx}$ • knows to put $\frac{dy}{dx}=0$ • finds x-coordinates • finds y-coordinates • determines nature of each by second derivative or nature table	• $\frac{dy}{dx}=1-\frac{1}{(x-1)^2}$ • $1-\frac{1}{(x-1)^2}=0$ • $x=0$ or $x=2$ • (0, -2), (2, 2) • $\frac{d^2y}{dx^2}=\frac{2}{(x-1)^3}$; $Max\ at\ (0,-2), Min\ at\ (2,2)$
13(d)	**ans:** sketch **4 marks** • sketch showing all relevant points • correctly shows how curve approaches asymptotes • knows to reflect all parts of graph from below the *x*-axis to above the *x*-axis • reflects correctly	See sketch at end of marking scheme
14(a)	**ans:** $\frac{4}{3}\pi a^3$ **8 marks** • draws sketch showing semi-circle above *x*-axis • Roots of semi-circle at $-a$ and a • knows how to find volume of revolution • limits of integration as $-a$ and a • applies formula correctly • integrates correctly • evaluates limits • correct answer	• and •  • and • $V=\int_{-a}^{a}\pi y^2dx$ • $V=\int_{-a}^{a}\pi\left(a^2-x^2\right)dx$ • $\pi\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a}$ • $\pi\left[a^2(a)-\frac{a^3}{3}\right]-\pi\left[a^2(-a)-\frac{(-a)^3}{3}\right]$ • $\frac{4}{3}\pi a^3$
14(b)	**ans:** 523·6 units^3 **2 marks** • knows to put $a=5$ • finds volume	• $\frac{4}{3}\pi\left(5^3\right)$ • 523·6 units^3

Total 100 Marks

Sketch for question 13(d)