ADVANCED HIGHER MATHEMATICS

FURTHER SEQUENCES AND SERIES

MACLAURIN SERIES

Consider the infinite geometric series $1 + x + x^2 + x^3 + \ldots$

If $-1 < x < 1$, this series converges and has a sum to infinity of $\dfrac{1}{1-x}$.

Hence $\quad 1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1-x}, \qquad -1 < x < 1,$

i.e. $\quad (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, \qquad -1 < x < 1.$

The function $f(x) = (1-x)^{-1}$ can therefore be expressed as an infinite series of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, where the coefficients a_0, a_1, a_2, a_3, ... are constants (provided $-1 < x < 1$).

In general, any function $f(x)$ can be expressed as an infinite series of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, where the coefficients a_0, a_1, a_2, a_3, ... are constants. This is known as expressing the function $f(x)$ as a **power series** or **Maclaurin series**. It is important to realise, however, that the power series may only converge for certain values of x.

Suppose in general that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots,$$

where the coefficients a_0, a_1, a_2, ... are constants.

The coefficients a_0, a_1, a_2, ... can be found by evaluating the function $f(x)$ and its derivatives at $x = 0$.

$$f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \ldots$$
$$\Rightarrow \quad f(0) = a_0$$
$$\Rightarrow \quad a_0 = f(0)$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$
$$\Rightarrow \quad f'(0) = a_1$$
$$\Rightarrow \quad a_1 = f'(0)$$

$$f''(x) = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \ldots$$

$$\Rightarrow \quad f''(0) = 2a_2$$

$$\Rightarrow \quad a_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4x + 5\cdot 4\cdot 3a_5x^2 + 6\cdot 5\cdot 4a_6x^3 + \ldots$$

$$\Rightarrow \quad f'''(0) = 3\cdot 2a_3$$

$$\Rightarrow \quad a_3 = \frac{f'''(0)}{3\cdot 2}$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5x + 6\cdot 5\cdot 4\cdot 3a_6x^2 + \ldots$$

$$\Rightarrow \quad f^{(4)}(0) = 4\cdot 3\cdot 2a_4$$

$$\Rightarrow \quad a_4 = \frac{f^{(4)}(0)}{4\cdot 3\cdot 2}$$

and so on.

Now $\quad f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots$

$$\Rightarrow \quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3\cdot 2}x^3 + \frac{f^{(4)}(0)}{4\cdot 3\cdot 2}x^4 + \ldots$$

Recall that $n!$ ("n factorial") denotes the product $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$ for any positive integer n ($0!$ is defined to be 1).

Hence:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

This result can be used to find the Maclaurin series for a given function $f(x)$.

Worked Example 1

Find the Maclaurin series for the function $f(x) = e^{2x}$ up to and including the term in x^4.

Solution

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2e^{2x} \cdot 2 = 4e^{2x}$

$f'''(x) = 4e^{2x} \cdot 2 = 8e^{2x}$

$f^{(4)}(x) = 8e^{2x} \cdot 2 = 16e^{2x}$

Hence: $f(0) = e^0 = 1$

$f'(0) = 2e^0 = 2$

$f''(0) = 4e^0 = 4$

$f'''(0) = 8e^0 = 8$

$f^{(4)}(0) = 16e^0 = 16$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad e^{2x} = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 + \ldots$$

$$\Rightarrow \quad e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots$$

[It can be shown that this series converges for all real values of x.]

Worked Example 2

Find the Maclaurin series for the function $f(x) = \cos 2x$ up to and including the term in x^4.

Solution

$f(x) = \cos 2x$

$f'(x) = -2\sin 2x$

$f''(x) = -2(2\cos 2x) = -4\cos 2x$

$f'''(x) = -4(-2\sin 2x) = 8\sin 2x$

$f^{(4)}(x) = 8(2\cos 2x) = 16\cos x$

Hence: $f(0) = \cos 0 = 1$

$f'(0) = -2\sin 0 = 0$

$f''(0) = -4\cos 0 = -4$

$f'''(0) = 8\sin 0 = 0$

$f^{(4)}(0) = 16\cos 0 = 16$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad \cos 2x = 1 + 0x + \frac{(-4)}{2}x^2 + \frac{0}{6}x^3 + \frac{16}{24}x^4 + \ldots$$

$$\Rightarrow \quad \cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + \ldots$$

[It can be shown that this series converges for all real values of x.]

Worked Example 3

Find the Maclaurin series for the function $f(x) = \ln(1+3x)$ up to and including the term in x^4.

Solution

$$f(x) = \ln(1+3x)$$

$$f'(x) = \frac{1}{1+3x} \cdot 3 = \frac{3}{1+3x} = 3(1+3x)^{-1}$$

$$f''(x) = -3(1+3x)^{-2} \cdot 3 = -9(1+3x)^{-2}$$

$$f'''(x) = 18(1+3x)^{-3} \cdot 3 = 54(1+3x)^{-3}$$

$$f^{(4)}(x) = -162(1+3x)^{-4} \cdot 3 = -486(1+3x)^{-4}$$

Hence: $f(0) = \ln 1 = 0$

$f'(0) = 3$

$f''(0) = -9$

$f'''(0) = 54$

$f^{(4)}(0) = -486$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad \ln(1+3x) = 0 + 3x + \frac{(-9)}{2}x^2 + \frac{54}{6}x^3 + \frac{(-486)}{24}x^4 + \ldots$$

$$\Rightarrow \quad \ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots$$

[It can be shown that this series only converges when x lies in the interval $-\frac{1}{3} < x \leq \frac{1}{3}$.]

Worked Example 4

Find the Maclaurin series for the function $f(x)=\sqrt{1+4x}$ up to and including the term in x^4.

Solution

$$f(x)=\sqrt{1+4x}=(1+4x)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{2}(1+4x)^{-\frac{1}{2}}\cdot 4=2(1+4x)^{-\frac{1}{2}}$$

$$f''(x)=-1(1+4x)^{-\frac{3}{2}}\cdot 4=-4(1+4x)^{-\frac{3}{2}}$$

$$f'''(x)=6(1+4x)^{-\frac{5}{2}}\cdot 4=24(1+4x)^{-\frac{5}{2}}$$

$$f^{(4)}(x)=-60(1+4x)^{-\frac{7}{2}}\cdot 4=-240(1+4x)^{-\frac{7}{2}}$$

Hence: $f(0)=\sqrt{1}=1$

$f'(0)=2$

$f''(0)=-4$

$f'''(0)=24$

$f^{(4)}(0)=-240$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\ldots$$

$$\Rightarrow \quad \sqrt{1+4x}=1+2x+\frac{(-4)}{2}x^2+\frac{24}{6}x^3+\frac{(-240)}{24}x^4+\ldots$$

$$\Rightarrow \quad \sqrt{1+4x}=1+2x-2x^2+4x^3-10x^4+\ldots$$

[It can be shown that this series only converges when x lies in the interval $-\frac{1}{4}<x<\frac{1}{4}$.]

Worked Example 5

(a) Obtain the Maclaurin series for the function $f(x) = e^x \sin x$ up to and including the term in x^4.

(b) **Hence** obtain the Maclaurin series for the function $g(x) = e^{2x} \sin 2x$ up to and including the term in x^4.

Solution

(a) $f(x) = e^x \sin x$

$$f'(x) = e^x \cdot \cos x + \sin x \cdot e^x \quad \text{[using the } \textbf{product rule}\text{]}$$
$$= e^x (\sin x + \cos x)$$

$$f''(x) = e^x (\cos x - \sin x) + (\sin x + \cos x) \cdot e^x \quad \text{[using the } \textbf{product rule}\text{]}$$
$$= e^x \{(\cos x - \sin x) + (\sin x + \cos x)\}$$
$$= e^x (2\cos x)$$
$$= 2e^x \cos x$$

$$f'''(x) = 2e^x \cdot (-\sin x) + \cos x \cdot 2e^x \quad \text{[using the } \textbf{product rule}\text{]}$$
$$= 2e^x (\cos x - \sin x)$$

$$f^{(4)}(x) = 2e^x \cdot (-\sin x - \cos x) + (\cos x - \sin x) \cdot 2e^x \quad \text{[using the } \textbf{product rule}\text{]}$$
$$= 2e^x \{(-\sin x - \cos x) + (\cos x - \sin x)\}$$
$$= 2e^x (-2\sin x)$$
$$= -4e^x \sin x$$

Hence: $f(0) = e^0 \sin 0 = 0$

$f'(0) = e^0 (\sin 0 + \cos 0) = 1$

$f''(0) = 2e^0 \cos 0 = 2$

$f'''(0) = 2e^0 (\cos 0 - \sin 0) = 2$

$f^{(4)}(0) = -4e^0 \sin 0 = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad e^x \sin x = 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 + \frac{0}{24}x^4 + \ldots$$

$$\Rightarrow \quad e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$

[*Alternatively, find the individual series for* e^x *and* $\sin x$ *and multiply these series together to find the series for* $e^x \sin x$.]

(b) $g(x) = e^{2x} \sin 2x$

We know that $e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$

Replacing x with $2x$ in this power series gives:

$$e^{2x} \sin 2x = 2x + (2x)^2 + \frac{1}{3}(2x)^3 + \ldots$$

$$\Rightarrow \quad e^{2x} \sin 2x = 2x + 4x^2 + \frac{1}{3} \cdot 8x^3 + \ldots$$

$$\Rightarrow \quad e^{2x} \sin 2x = 2x + 4x^2 + \frac{8}{3}x^3 + \ldots$$

Worked Example 6

(a) Obtain the Maclaurin series for $\sin^2 x$ up to an including the term in x^4.

(b) **Hence** obtain the Maclaurin series for $\cos^2 x$ up to and including the term in x^4.

Solution

(a) $f(x) = \sin^2 x = (\sin x)^2$

$f'(x) = 2\sin x \cdot \cos x$

$$f''(x) = 2\sin x \cdot (-\sin x) + \cos x \cdot 2\cos x \quad \text{[using the \textbf{product rule}]}$$
$$= 2\cos^2 x - 2\sin^2 x$$
$$= 2(\cos x)^2 - 2(\sin x)^2$$

$$f'''(x) = 4\cos x \cdot (-\sin x) - 4\sin x \cdot \cos x$$
$$= -4\sin x\cos x - 4\sin x\cos x$$
$$= -8\sin x\cos x$$

$$f^{(4)}(x) = -8\sin x \cdot (-\sin x) + \cos x \cdot (-8\cos x) \quad \text{[using the \textbf{product rule}]}$$
$$= 8\sin^2 x - 8\cos^2 x$$

Hence: $f(0) = \sin^2 0 = 0$

$f'(0) = 2\sin 0\cos 0 = 0$

$f''(0) = 2\cos^2 0 - 2\sin^2 0 = 2$

$f'''(0) = -8\sin 0\cos 0 = 0$

$f^{(4)}(0) = 8\sin^2 0 - 8\cos^2 0 = -8$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

$$\Rightarrow \quad \sin^2 x = 0 + 0x + \frac{2}{2}x^2 + \frac{0}{6}x^3 + \frac{(-8)}{24}x^4 + \dots$$

$$\Rightarrow \quad \sin^2 x = x^2 - \frac{1}{3}x^4 + \dots$$

[Note that $f'(x) = 2\sin x\cos x = \sin 2x$.
This makes it easier to find the higher derivatives:

$$f''(x) = 2\cos 2x$$
$$f'''(x) = 2(-2\sin 2x) = -4\sin 2x$$
$$f^{(4)}(x) = -4(2\cos 2x) = -8\cos 2x\ ]$$

[*Alternative Method: Note that* $\sin^2 x = \sin x \sin x$ *and use the Maclaurin series for* $\sin x$ *to find the series for* $\sin^2 x$.]

(b) $\cos^2 x = 1 - \sin^2 x$

$$= 1 - \left(x^2 - \frac{1}{3}x^4 + \ldots\right)$$

$$= 1 - x^2 + \frac{1}{3}x^4 + \ldots$$

Worked Example 7

Obtain the Maclaurin series for the function $f(x) = \tan^{-1} x$ up to and including the term in x^3.

Solution

$f(x) = \tan^{-1} x$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$\begin{aligned} f''(x) &= -(1+x^2)^{-2} \cdot 2x \\ &= \frac{-2x}{(1+x^2)^2} \end{aligned}$$

$$f'''(x) = \frac{(1+x^2)^2 \cdot \frac{d}{dx}(-2x) - (-2x) \cdot \frac{d}{dx}(1+x^2)^2}{\left\{(1+x^2)^2\right\}^2} \quad \text{[using the \textbf{quotient rule}]}$$

$$\begin{aligned} &= \frac{(1+x^2)^2 \cdot (-2) + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} \\ &= \frac{8x^2(1+x^2) - 2(1+x^2)^2}{(1+x^2)^4} \\ &= \frac{2(1+x^2)\left\{4x^2 - (1+x^2)\right\}}{(1+x^2)^4} \\ &= \frac{2(1+x^2)(3x^2-1)}{(1+x^2)^4} \\ &= \frac{2(3x^2-1)}{(1+x^4)^3} \end{aligned}$$

Hence: $f(0) = \tan^{-1} 0 = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -2$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad \tan^{-1} x = 0 + 1x + \frac{0}{2}x^2 + \frac{(-2)}{6}x^3 + \ldots$$

$$\Rightarrow \quad \tan^{-1} x = x - \frac{1}{3}x^3 + \ldots$$

THE FIXED POINTS OF A RECURRENCE RELATION

Consider the linear recurrence relation

$$x_{n+1} = 0 \cdot 2x_n + 4$$

with a given starting value x_0.

The sequence of values x_0, x_1, x_2, ... generated by this recurrence relation will converge to a limit L since $-1 < 0 \cdot 2 < 1$.

The limit L satisfies the equation

$$L = 0 \cdot 2L + 4,$$

since $x_n \to L$ and $x_{n+1} \to L$ as $n \to \infty$.

This equation can be solved to give $L = 5$, hence the limit is 5.

Now consider the recurrence relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

with a given starting value x_0.

Assuming that a limit exists as $n \to \infty$, the limit L will satisfy the equation

$$\begin{aligned}
& L = \frac{1}{2}\left(L + \frac{5}{L}\right) \quad [\times 2] \\
\Rightarrow \quad & 2L = L + \frac{5}{L} \quad [\times L] \\
\Rightarrow \quad & 2L^2 = L^2 + 5 \\
\Rightarrow \quad & L^2 = 5 \\
\Rightarrow \quad & L = \pm\sqrt{5}
\end{aligned}$$

The sequence of values x_0, x_1, x_2, ... generated by this recurrence relation converges to one of two different limits, depending on the starting value x_0.

In broad terms, if the starting value x_0 is "close" to $\sqrt{5}$, then the sequence of values x_0, x_1, x_2, ... will converge to the limit $\sqrt{5}$.

On the other hand, if the starting value x_0 is "close" to $-\sqrt{5}$, then the sequence of values x_0, x_1, x_2, ... will converge to the limit $-\sqrt{5}$.

So if $x_0 = 2$, the sequence will converge to the limit $\sqrt{5}$, whereas if $x_0 = -2$, the sequence will converge to the limit $-\sqrt{5}$.

In this context, the limits $\sqrt{5}$ and $-\sqrt{5}$ are known as the **fixed points** of the recurrence relation.

FINDING AN APPROXIMATE ROOT OF THE EQUATION $x = f(x)$

An approximate root of the equation $x = f(x)$ can sometimes be found using the recurrence relation

$$x_{n+1} = f(x_n),$$

where the starting value x_0 is an initial approximation to the root.

Using the recurrence relation $x_{n+1} = f(x_n)$, if the sequence of values x_0, x_1, x_2, ... converges to a limit L, then it can be shown that the limit L is in fact a root of the equation $x = f(x)$.

[If it exists, the limit L of the sequence generated by the recurrence relation $x_{n+1} = f(x_n)$ will satisfy the equation $L = f(L)$ since $x_n \to L$ and $x_{n+1} \to L$ as $n \to \infty$. Thus L is a root of the equation $x = f(x)$.]

Worked Example

The equation $x^3 - 2x - 1 = 0$ has a root in the interval $1 < x < 3$.

(a) Verify that the equation can be rewritten as $x = (2x+1)^{\frac{1}{3}}$.

(b) By using the iterative scheme $x_{n+1} = (2x_n + 1)^{\frac{1}{3}}$ with $x_0 = 2$, obtain an approximation to the root which is correct to two decimal places.

Solution

(a) $x^3 - 2x - 1 = 0 \quad \Rightarrow \quad x^3 = 2x + 1$

$\Rightarrow \quad x = (2x+1)^{\frac{1}{3}}$

(b) $x_{n+1} = (2x_n + 1)^{\frac{1}{3}}$

$x_0 = 2$

$x_1 = (2x_0 + 1)^{\frac{1}{3}} = 5^{\frac{1}{3}} = 1 \cdot 7099...$

$x_2 = 1 \cdot 6411...$

$x_3 = 1 \cdot 6238...$

$x_4 = 1 \cdot 6195...$

$x_5 = 1 \cdot 6184...$

$x_6 = 1 \cdot 6181...$

$x_7 = 1 \cdot 6180...$

$x_8 = 1 \cdot 6180...$

The root is $x = 1 \cdot 62$ correct to two decimal places.

[*Note 1: Do not round off the value of* x_n *when used to calculate the value of* x_{n+1}. *It is recommended that a graphics calculator is used to generate the values of the sequence. Alternatively, the memory facility on a scientific calculator can be used to store the full value of* x_n *before calculating the value of* x_{n+1}.]

[*Note 2: The equation* $x^3 - 2x - 1 = 0$ *can also be rewritten as* $x = \frac{1}{2}(x^3 - 1)$. *This suggests using the iterative scheme* $x_{n+1} = \frac{1}{2}(x_n{}^3 - 1)$ *to obtain an approximation to the root. You can verify, however, that the values generated by this recurrence relation do not converge to a limit. When using this method to find an approximate root of an equation, care must therefore be taken when rewriting the equation in the form* $x = f(x)$. *There is often more than one possible rearrangement and not all will be useful. If one rearrangement does not prove useful, you should always look to try a different rearrangement.*]